Theoretical Challenges in Wave Turbulence, 8-9 May,
University of Warwick

Spectral evolution of weakly nonlinear random wave fields: kinetic equations vs direct numerical simulations

What is wrong with the kinetic equations?

Victor Shrira & Sergei Annenkov

Keele University, UK

Motivation

Wave fields in nature (and, in seas, in particular) are intrinsically random. They can be described deterministically only for relatively short spans (For water waves $\sim 10^2$ periods (Annenkov & Shrira 2001)).

The challenge of describing wave turbulence, i.e. evolution of random weakly nonlinear dispersive waves in various contexts is a major open fundamental problem.

Most existing theoretical approaches to modelling of long-term evolution of random weakly nonlinear waves are based on the wave kinetic equation (KE). But, **we do not know how good it is.**

Here we examine the kinetic equation (KE) and its generalization by comparing their predictions with direct numerical simulations. We show that there is a significant discrepancy and, therefore, **the fundamental assumptions underpinning the KEs have to be re-visited.**

Background

We will be interested in spatially uniform media where all weakly nonlinear waves without loss of generality can be described as an ensemble of interacting Fourier modes (w.r.t. spatial coordinates) characterized by their time dependent complex amplitudes $a(\mathbf{k}, t)$. Long time evolution of such modes is intrinsically random and therefore for large times only a statistical description makes sense. Hence the ultimate goal of all efforts aimed at understanding random nonlinear waves in a myriad of physical contexts (plasmas, ocean surface waves, internal gravity and Rossby waves in the ocean and atmosphere, waves in solid body shells, spin waves in magnetics, etc) is to find how statistical characteristics of the wave field at hand evolve. The full description of a random field should yield time dependence of probability density function and of all kind of correlators. At present we are very far away from that goal. The most common approach is to introduce a hierarchy of momenta

$$\langle a_{\mathbf{k}} a^*_{\mathbf{k}1} \rangle, \ \langle a_{\mathbf{k}1} a^*_{\mathbf{k}2} a^*_{\mathbf{k}3} \rangle, \ ...$$

and try to find out their evolution.

Most of the theoretical and experimental efforts was concentrated on the evolution of the second moment. The equations governing the evolution of the second moment (i.e., the energy spectra) are called **kinetic equations (KEs)**.

When we consider weakly nonlinear waves, then, to leading order statistical properties do not evolve, since the linear modes do not interact. Employing standard perturbation approach one can derive an infinite chain of equations for the moments: the evolution of the second order moments is expressed in terms of the the third or fourth order moments, the latter are expressed in terms of higher order moments and so forth. To truncate this infinite chain of equations the only way practised universally so far is to employ **"closure hypothesis"**.

Here we do not report progress in this direction, on the contrary, we will show that the universally adopted closure hypothesis has to be re-visited.

In the water wave context the KE is often referred to as the Hasselmann equation (Hasselmann 1962).

$$\frac{dn(\mathbf{k},\mathbf{x},t)}{dt} = S_{input} + S_{diss} + S_{nl}$$

where $\langle a_{\mathbf{k}} a^*_{\mathbf{k}1} \rangle = n(\mathbf{k},t)\delta(\mathbf{k}-\mathbf{k}_1)$, $n(\mathbf{k}$ is the 2D wave action spectrum. The interaction term S_{nl}, dominant for energy carrying waves, is derived from first principles employing an asymptotic procedure based upon smallness of nonlinearity parameter ε and a number of additional assumptions:

$$S_{nl} = 4\pi \int T^2_{0123} f_{0123} \delta_{0+1-2-3} \delta(\omega_0 + \omega_1 - \omega_2 - \omega_3)\, \mathrm{d}\mathbf{k}_{123}, \quad (1)$$

where $f_{0123} = n_2 n_3 (n_0 + n_1) - n_0 n_1 (n_2 + n_3)$, $n_i \equiv n(\mathbf{k}_i)$, $\delta_{0+1-2-3} \equiv \delta(\mathbf{k}_0 + \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3)$ and T_{0123} is given by an explicit but a page long expression.

Background

The KE is based on two major assumptions:

- **quasi-gaussianity** (the central element of the statistical closure)
- **quasi-stationarity**

textbfemphQuasi-stationarity means that the KE is not applicable to the situations with rapid changes of the environment, such as wind gusts. Due to lack of alternatives, this fact is usually ignored, and the standard KE is used to model the response to an instant and sharp increase or decrease of wind (e.g. , Young & van Agthoven 1997).

Quasi-gaussianity is the central assumption:the phases of interacting waves are assumed to be totally random to leading order, weak correlations emerge due to nonlinearity.

A clarification of degree of validity and the role of both assumptions is important and relevant within and beyond the water wave context.

Generalised kinetic equation (gKE)

The gKE is derived using the same statistical closure as the KE, but without the assumption of quasi-stationarity. In the derivation of the kinetic theory, we have the equation for the spectrum in terms of the higher-order cumulant $J^{(1)}_{0123}$: $J_{0123}\delta_{0+1-2-3} =< b_0 b_1 b_2^* b_3^* >$

$$\frac{\partial n_0}{\partial t} = 2\mathrm{Im}\int T_{0123} J^{(1)}_{0123}\delta_{0+1-2-3}\,\mathrm{d}\mathbf{k}_{123},$$

and the equation for the cumulant

$$\left(\mathrm{i}\frac{\partial}{\partial t} + \Delta\omega\right) J^{(1)}_{0123} = -2T_{0123} f_{0123},$$

where $\Delta\omega = \omega_0 + \omega_1 - \omega_2 - \omega_3$, $f_{0123} = n_2 n_3 (n_0 + n_1) - n_0 n_1 (n_2 + n_3)$. Classic KE derivation drops $\partial/\partial t$ and leads to the approximate solution for large time in terms of generalised functions

$$J^{(1)}_{0123}(t) = -2T_{0123}\left[\frac{P}{\Delta\omega} + \mathrm{i}\pi\delta(\Delta\omega)\right] f_{0123}(t),$$

where P is "principal value", δ is Dirac δ-function.

Generalised kinetic equation (gKE)

The gKE is derived using the exact solution of the differential equation for the cumulant (Annenkov & Shrira 2006 JFM, 561). The resulting equation (gKE) has the form

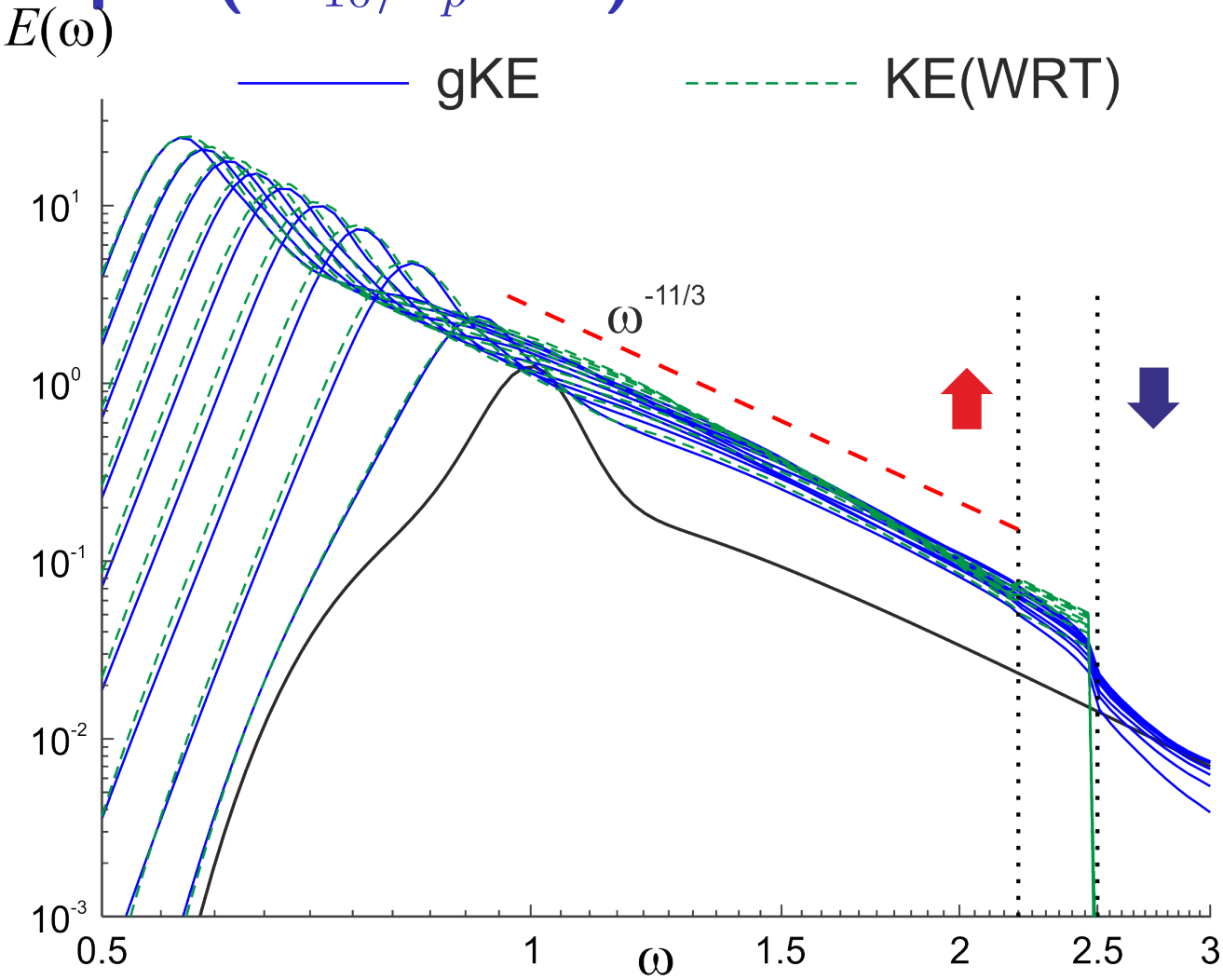
$$\frac{\partial n_0}{\partial t} = 4\mathrm{Re}\int\left\{T^2_{0123}\left[\int_0^t \mathrm{e}^{-\mathrm{i}\Delta\omega(\tau-t)} f_{0123}\,\mathrm{d}\tau\right] - \frac{\mathrm{i}}{2}T_{0123}J^{(1)}_{0123}(0)\mathrm{e}^{\mathrm{i}\Delta\omega t}\right\}\delta_{0+1-2-3}\,\mathrm{d}\mathbf{k}_{123} + S_{inp/diss}\,.$$

The gKE is nonlocal in time: evolution of the spectrum depends on the previous history of evolution, starting from the initial moment when the value of cumulant $J^{(1)}_{0123}(0)$ is prescribed as the initial condition. However, the gKE can be solved iteratively. On each time step, the value of $J^{(1)}_{0123}$ is computed and taken as the new initial condition, so that the 'internal' time integration is performed over one timestep only.
Details of the algorithm: Annenkov & Shrira, Modelling transient sea states with the generalised kinetic equation, In: *Rogue and Shock Waves in Nonlinear Dispersive Media*, M.Onorato et al (eds), Springer, 2016.

Generalised kinetic equation (gKE) - numerics

- specify a computational grid $\omega_{min} \leq \omega \leq \omega_{max}$ and $\theta_{min} \leq \theta \leq \theta_{max}$, ω is spaced logarithmically
- for each three grid points a fourth wave is found as $\mathbf{k}_4 = \mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3$, and bilinear interpolation is used to find the corresponding amplitude
- both resonant and non-resonant interactions are taken into account, with a certain (large) cutoff
- all interaction coefficients are pre-calculated, stored and distributed evenly across the parallel processors
- initial condition for $J^{(1)}_{0123}(0)$ is taken as zero ("cold start")
- the right-hand side of the gKE and the value of $J^{(1)}_{0123}$ are computed on each timestep, and $J^{(1)}_{0123}$ is used as the new initial condition
- standard Runge-Kutta-Fehlberg time-stepping algorithm with automatic step choice
- since almost all computations are performed in parallel, the algorithm has a nearly perfect scalability

Example ($U_{10}/c_p = 5$)



Evolution of the energy spectrum $E(\omega)$ with time, under constant wind forcing with initial $U/c_p = 5$. Spectra are plotted every 100 characteristic periods, gKE (blue curves) vs KE (dashed green curves, WRT(Webb-Resio-Tracey) code kindly provided by Gerbrandt van Vledder)

Direct numerical simulation (DNS-ZE)

Based on the Zakharov equation

$$\mathrm{i}\frac{\partial b_0}{\partial t} = \omega_0 b_0 + \int T_{0123} b_1^* b_2 b_3 \delta_{0+1-2-3}\, \mathrm{d}\mathbf{k}_{123} + \dots$$

If we want to apply a dynamical algorithm for the study of wave statistics, we have to overcome one substantial difficulty. For numerics, a continuous wave field needs to be discretised, i.e.

$$b(\mathbf{k}, t) = \sum_{j=1}^{N} b_j(\mathbf{k}_j, t).$$

Most models of nonlinear evolution employ a fast Fourier transform on each step, which requires a regular grid. The Zakharov equation allows the use of an arbitrary grid. However, for any grid the resulting discrete wave system will have properties differing from those of a continuous wave field. In order to model a continuous wave field correctly, every degree of freedom of a discretised wave field is expected to interact with every other degree of freedom. This means that we need, instead of a straightforward discretisation, to work out the concept of *coarse-graining* of the continuous wave field, which would retain its fundamental properties of nonlinear interactions.

We build in Fourier space a grid consisting of $\sim 5 \cdot 10^3$ wave packets, coupled through exact and approximate resonant interactions. A wave packet, centred at $\mathbf{k}_0$, is characterised by one amplitude and one phase, but has finite bandwidth in Fourier space, and is allowed to enter into nonlinear interactions with other wavepackets, provided that the wavevector mismatch

$$\Delta \mathbf{k} = \mathbf{k}_0 + \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3$$

does not exceed a certain threshold (the coarse-graining parameter). Thus, the standard resonance condition $\mathbf{k}_0 + \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3 = 0$ is relaxed. It has been verified that we need to consider only resonant and approximately resonant interactions, prescribing a similar condition on the frequency mismatch $\Delta\omega$, where

$$\Delta\omega = \omega_0 + \omega_1 - \omega_2 - \omega_3.$$

In more practical terms, the following condition is formulated: a quartet of grid points is assumed to be in approximate resonance if its wavevector and frequency mismatch satisfies

$$\Delta\omega/\omega_{min} < \lambda_\omega, \qquad |\Delta\mathbf{k}|/k_{min} < \lambda_k\bar{\omega}/\omega_{min}, \tag{2}$$

where $\Delta\omega$ and $|\Delta\mathbf{k}|$ are the frequency and wavevector mismatch in the quartet, $\omega_{\min}$ and $k_{\min}$ are the minimum values of frequency and wavenumber in the quartet, $\bar{\omega}$ is the mean frequency, and λ_ω and λ_k are the detuning parameters, chosen to ensure that the total number of resonances is $O(N^2)$, where N is the number of grid points. The resulting system of N discrete equations can be integrated in time by a standard Runge-Kutta scheme. Here, we will use 161 $\mathbf{k}$-points $9 < k < 355$ m^{-1} and 41 θ-points within $-4\pi/9 \le \theta \le 4\pi/9$, and $\lambda_k = 0.03$, $\lambda_\omega = 0.01$. The total number of resonant and near-resonant interactions is approximately $3.2 \cdot 10^8$. Initial phases of waves are random, averaging is over 100 realisations.

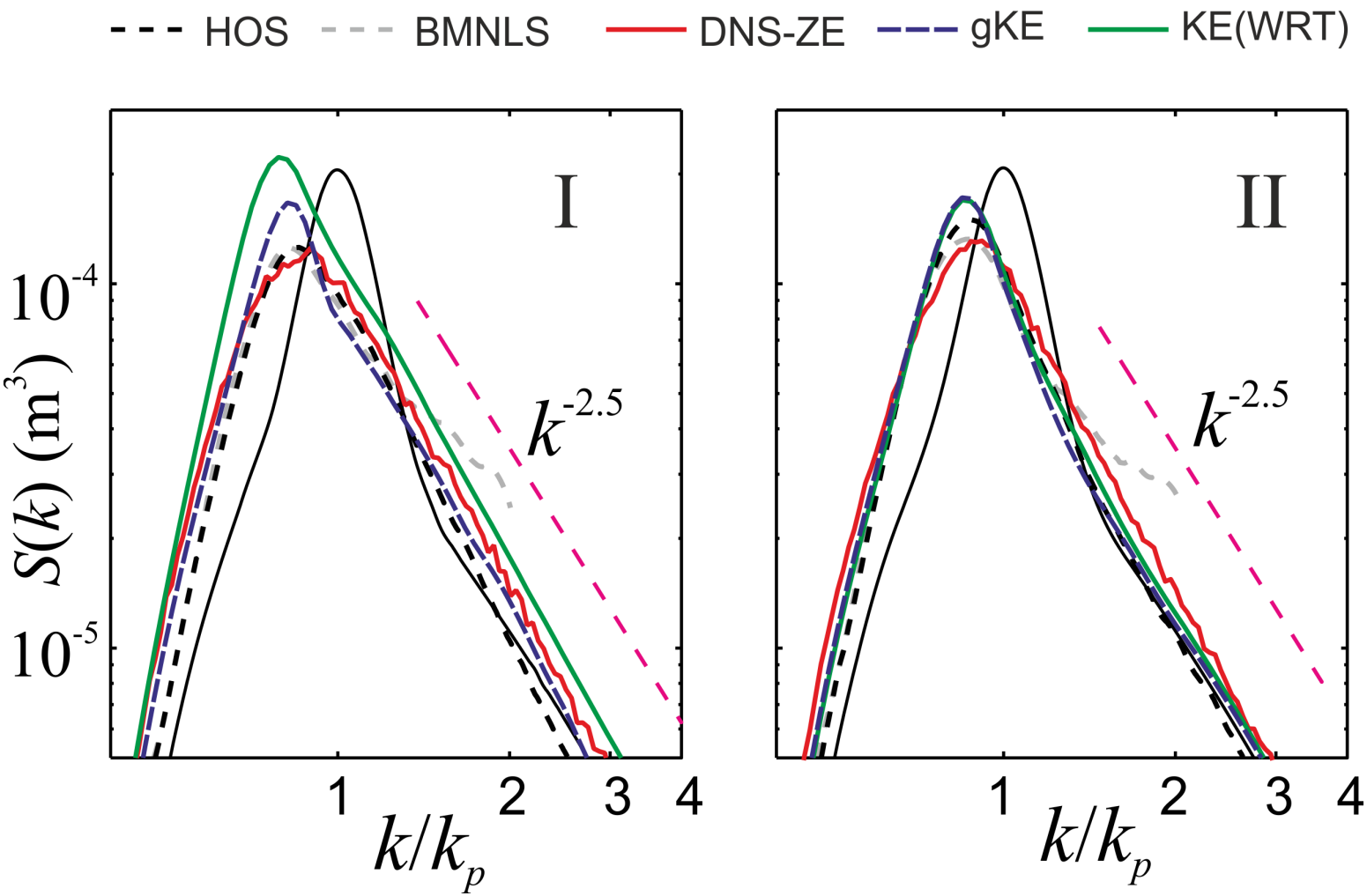
Initial conditions

We are going to compare all three approaches (KE, gKE and DNS-ZE) and some other simulations of certain initial spectra, without wind forcing. As initial conditions, we consider two JONSWAP spectra with the same frequency distribution ($H_s = 0.08$ m, $T_p = 1$ s, $\varepsilon = 0.11$, and $\gamma = 6$), different only in the initial directional distribution.

- Spectrum I ("narrow") – corresponds to $N = 840$ in the $\cos^N$ model
- Spectrum II ("wide") – corresponds to $N = 24$

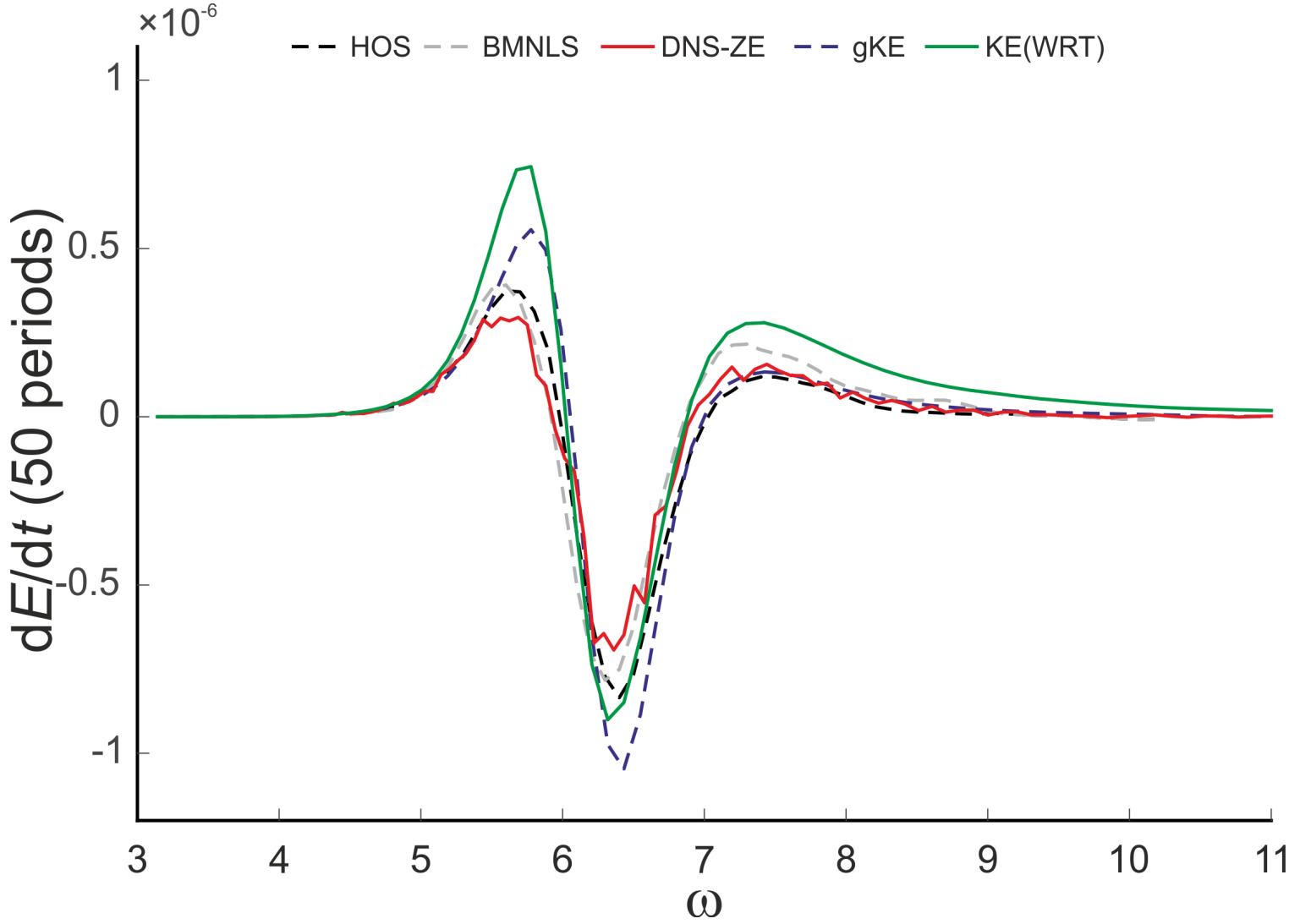
The same spectra were used as initial conditions in the experimental study by Onorato et al (2009) and numerical studies by Toffoli et al (2010) and Xiao et al (2013).
In particular, Xiao et al (2013) performed numerical simulations of the evolution (only about 150 periods) of the same initial spectra using higher-order spectral method (HOS) and broadband NLS (Dysthe equation, BMNLS).
Thus, we can consider the short-term evolution of these spectra (without wind forcing) with five different approaches, based on different sets of assumptions, and use the results for comparison and validation of the new algorithms.

first 150 periods



Evolution (first 150 periods) of spectrum I (narrow in angle) and II (wider in angle), with a direct comparison of 5 approaches (modified from figure 7*a*,*b* of Xiao et al 2013)

Growth rates over first 50 periods, spectrum I



Growth rates $\mathrm{d}E(\omega,t)/\mathrm{d}t$ over first 50 periods of evolution, with 5 approaches (values for HOS and BMNLS taken from figure 7 of Xiao et al 2013). Initial peak is at $\omega = 2\pi$

Directional spread

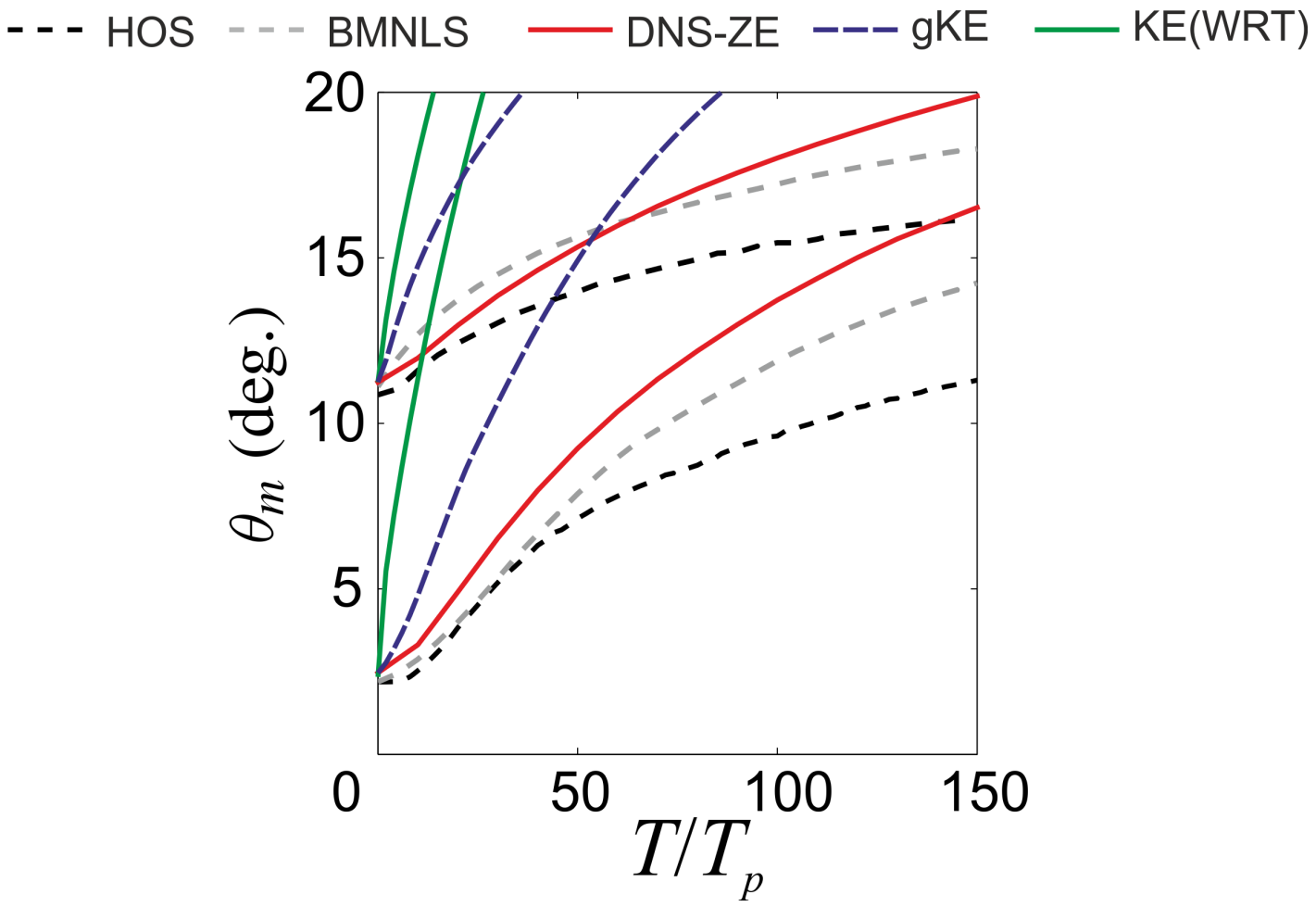
As a measure of the angular width of the spectrum, it is convenient to use the average of the second-order moment of directional distribution, defined as

$$\theta_m = \overline{\theta_2(k)},$$

$$\theta_2(k) = \left(\int_0^{\pi/2} \theta^2 D(k,\theta)\,\mathrm{d}\theta \right)^{1/2} \left(\int_0^{\pi/2} D(k,\theta)\,\mathrm{d}\theta \right)^{-1/2},$$

where $D(k,\theta)$ is the angular distribution function of the spectrum (Hwang *et al* 2000).

Evolution of mean directional width



Evolution (first 150 periods) of the averaged angular spread θ_m of spectra I and II, with a direct comparison of 5 approaches (modified from figure 7*c* of Xiao et al 2013)

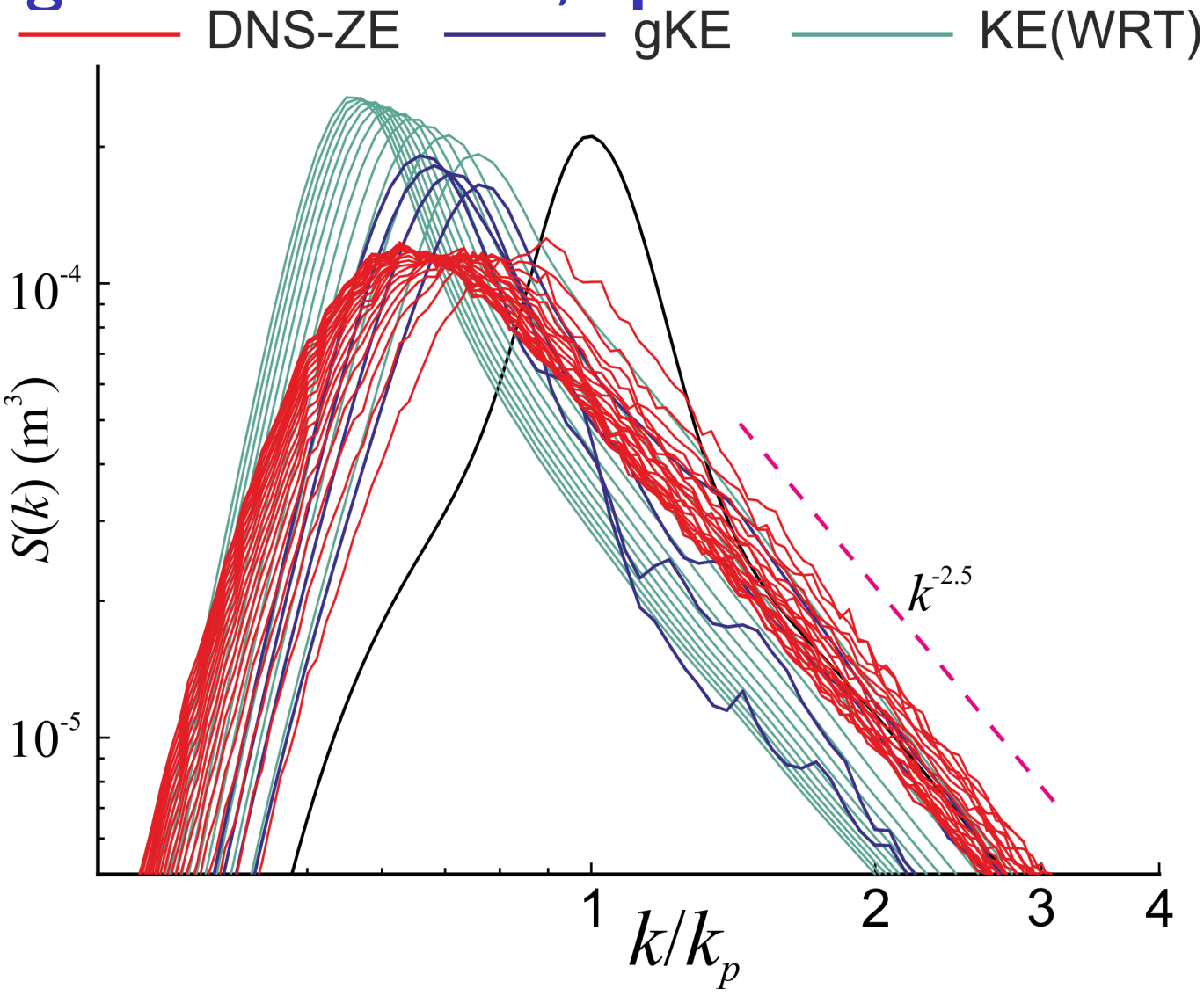
Short-term evolution – summary

Direct comparison of DNS-ZE with HOS and BMNLS, and of the two kinetic equations shows that

- KE and gKE results coincide in the wider case II
- in the narrow case I, the KE overestimates the amplitude of the spectral peak, compared to gKE
- DNS-ZE, HOS and BMNLS are consistent with each other, but different from both kinetic equations
- the kinetic equations show more narrow spectra, with a pronounced overshoot, while the DNS algorithms give wider spectra with lower amplitude of the peak
- there is a dramatic difference in the rate of angular broadening, which is consistent between DNS-ZE, HOS and BMNLS, much higher for gKE, and even higher for the KE
- growth rates over the first 50 periods are higher for the kinetic equations than for the DNS algorithms

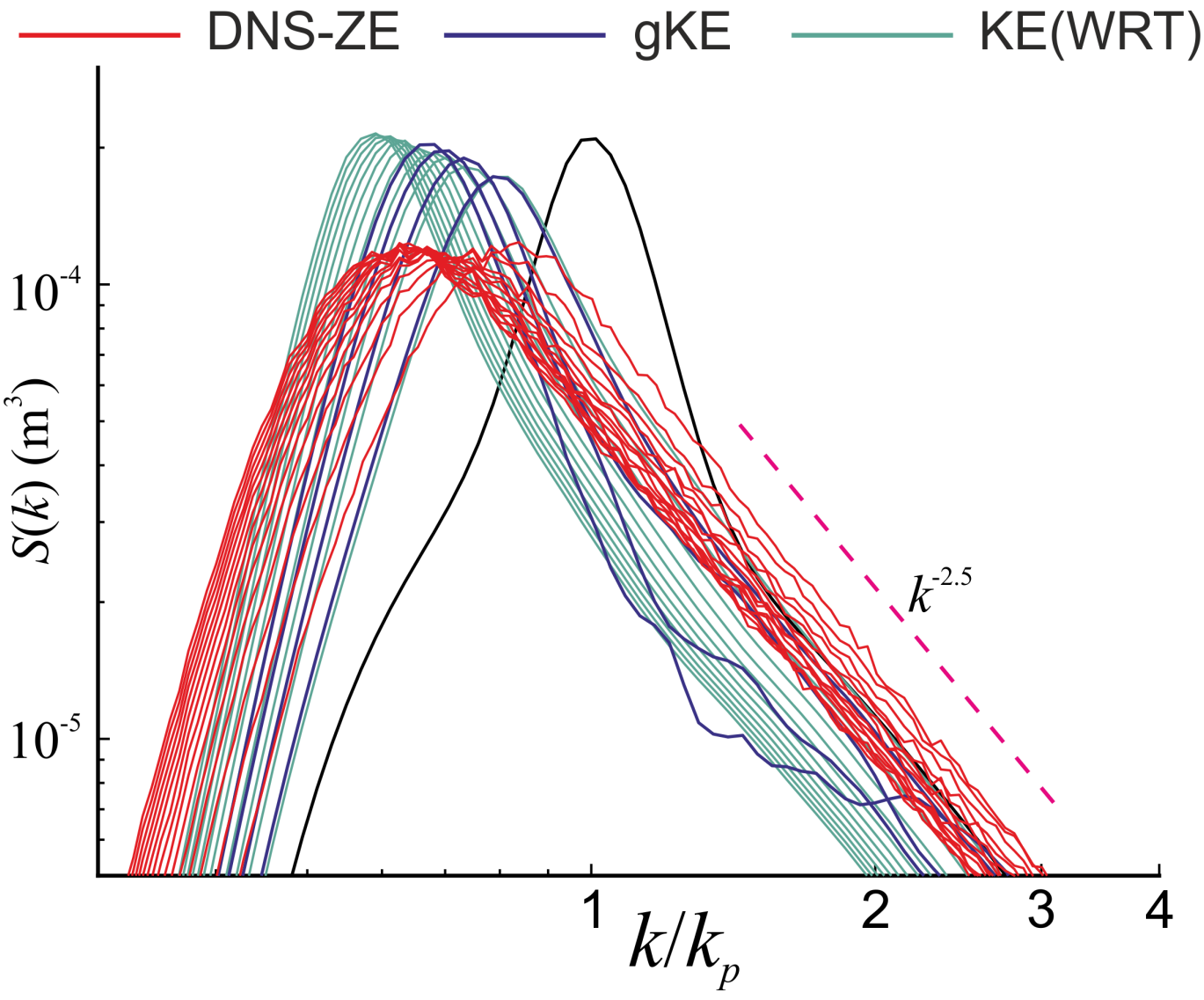
This validates both the gKE and the DNS-ZE approaches in the short term. Now we can proceed with studying the long-term evolution

Long-term evolution, spectrum I



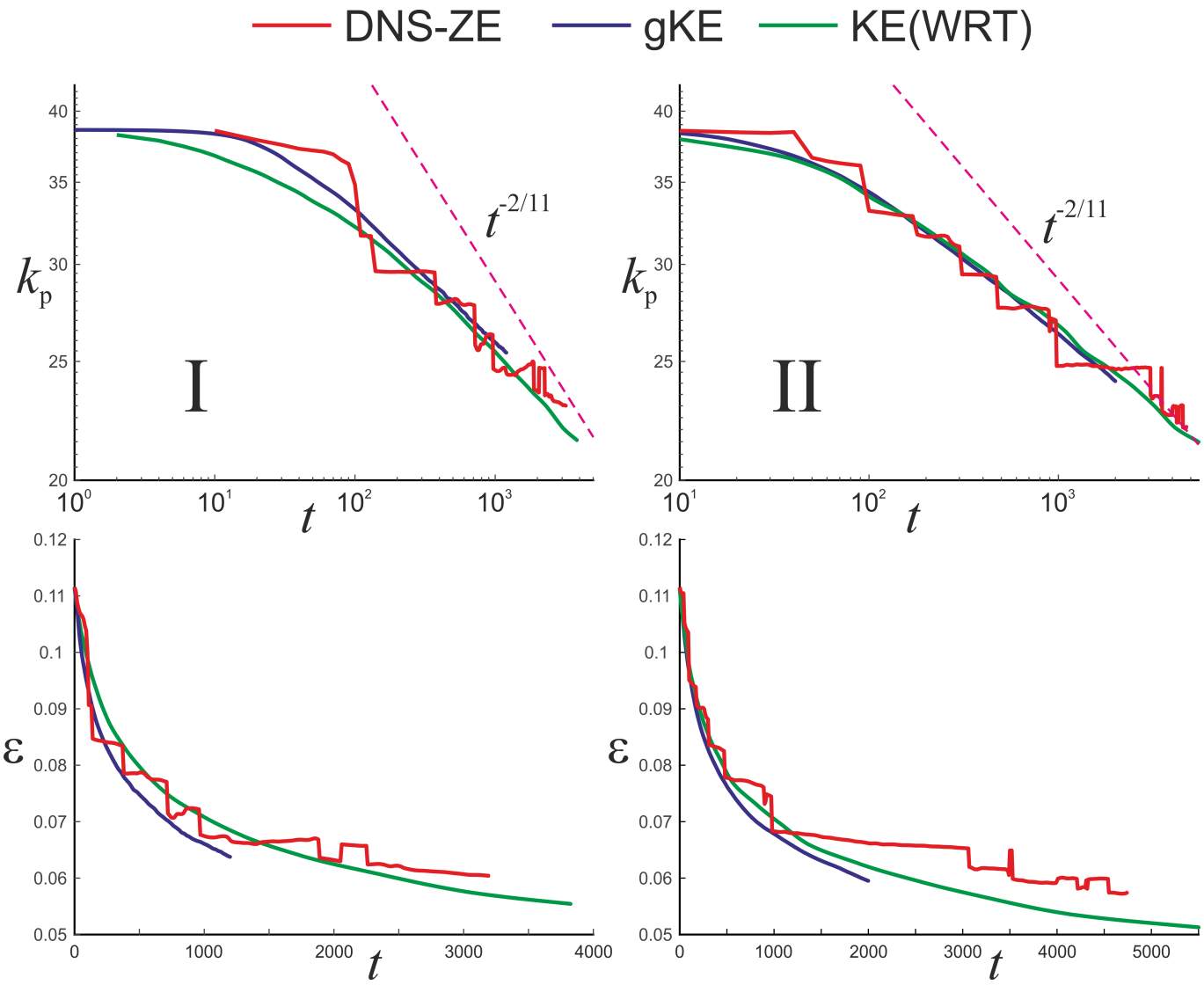
Long-term spectral evolution for spectrum I, with the comparison of DNS-ZE and both kinetic equations (KE and gKE). Spectra are plotted every 300 periods

long-term evolution, spectrum II



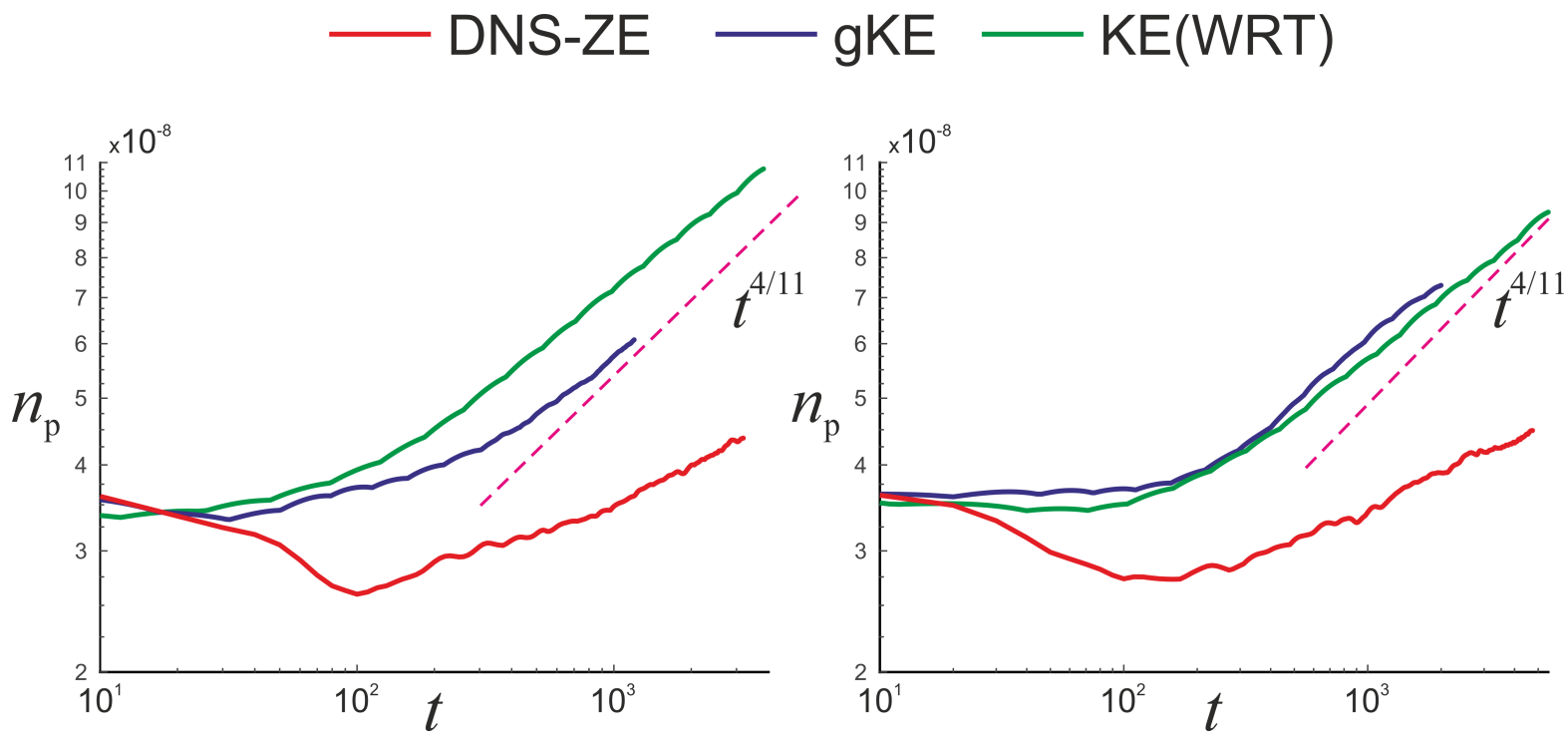
Long-term spectral evolution for spectrum II

Peak wavenumber and wave steepness



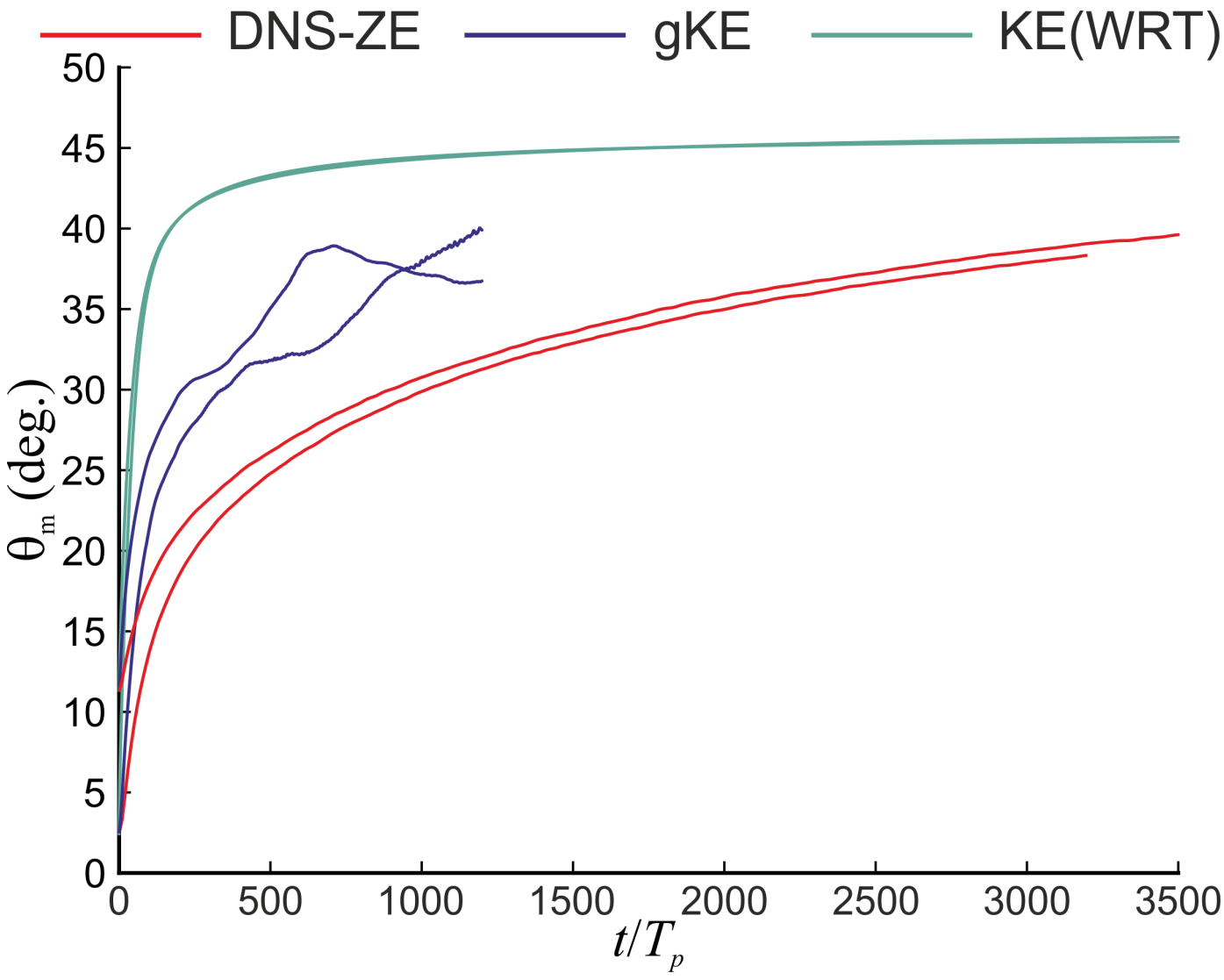
Evolution of the wavenumber of the spectral peak (vs the theoretical asymptotics: $\sim t^{-2/11}$) and wave steepness for spectra I and II

Peak amplitude

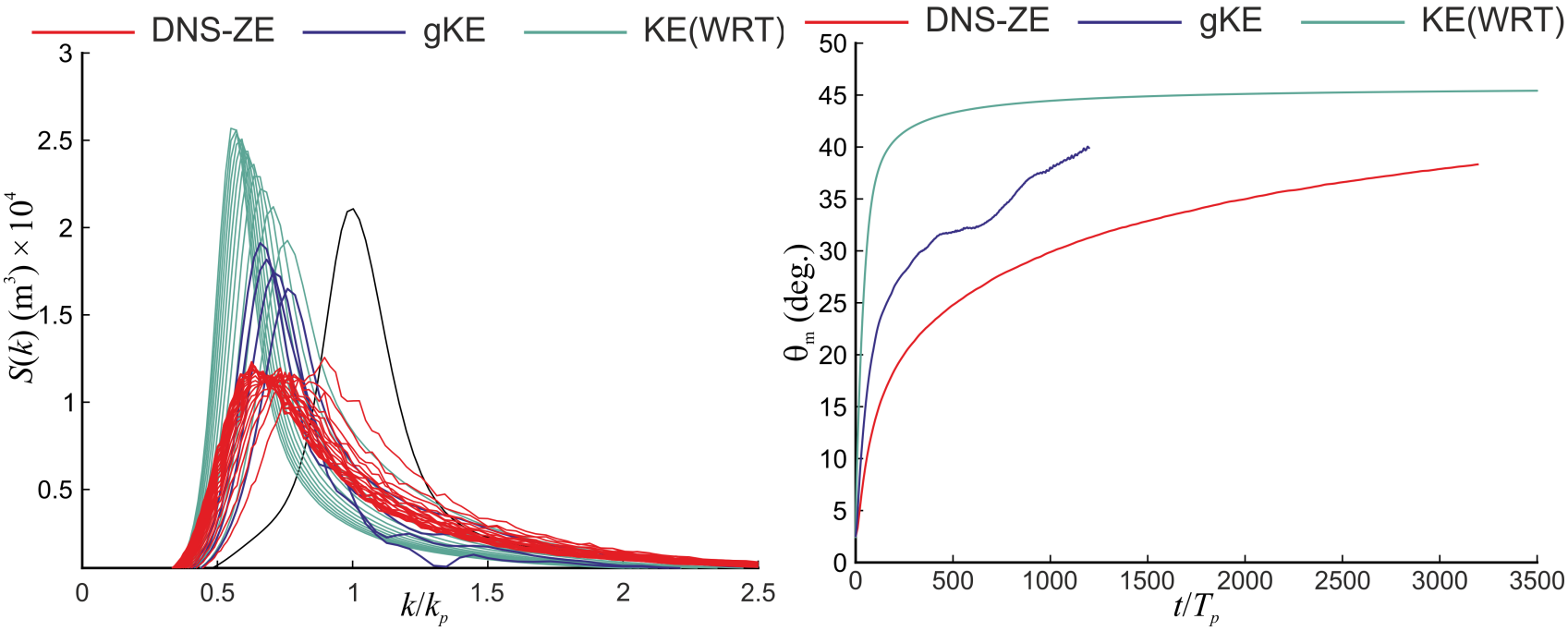


Evolution of the amplitude of the spectral peak (with theoretical asymptotic $\sim t^{4/11}$) for spectra I and II

long-term evolution of mean directional spreading



Evolution of a narrow initial spectrum again



Evolution of initial spectrum I: direct comparison of DNS-ZE, KE and gKE. Spectra are plotted every 300 periods.

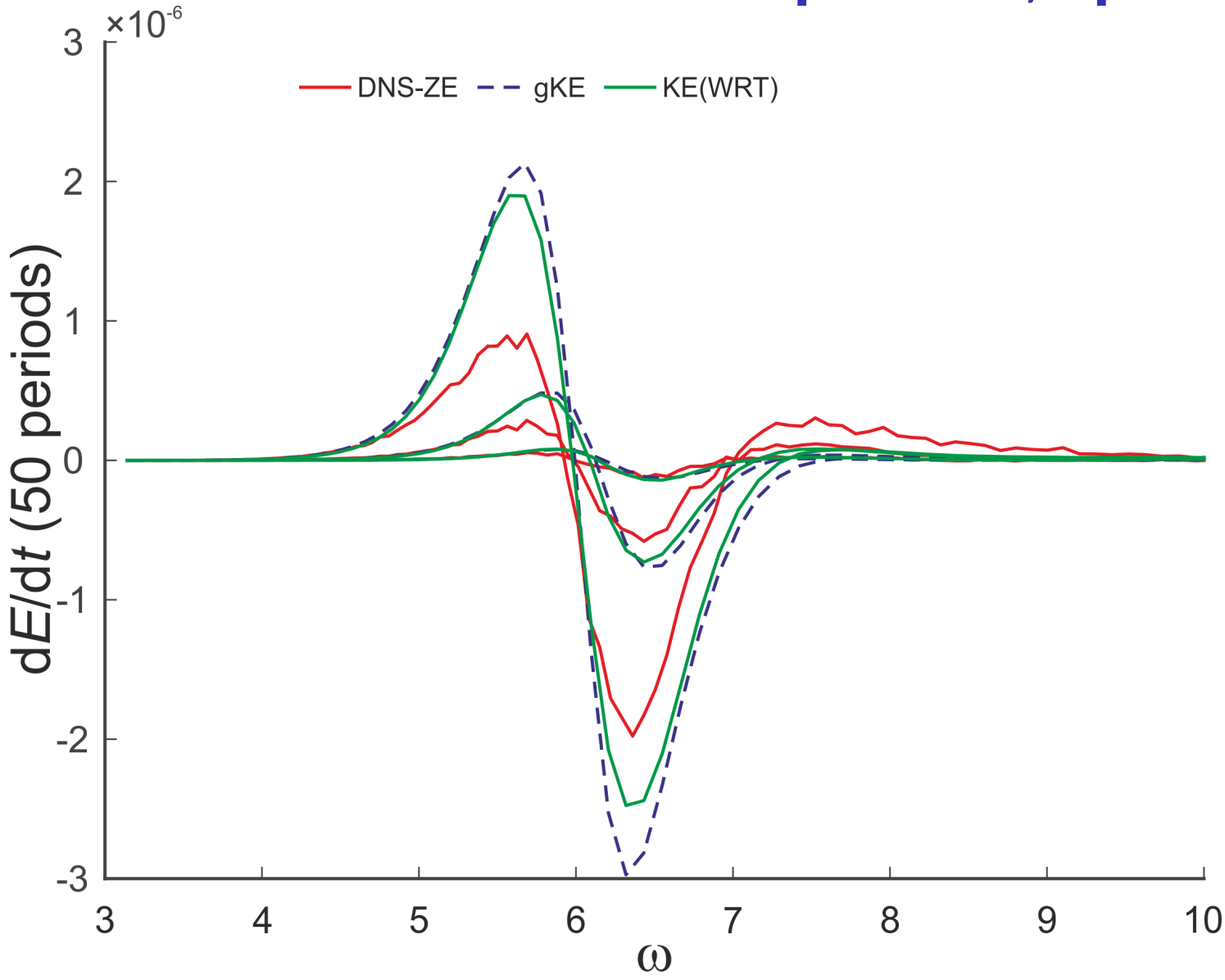
Role of modulational instability

Is the modulational instability responsible for the difference between DNS and the kinetic equations spectra?

- The modulational instability is part of the dynamics, but is not present in the statistical models.
- In order to clarify its role, we consider a "modified" Zakharov equation with the opposite sign of the interaction coefficient.
- This equation leads to the same kinetic theory (which does not depend on the sign of the coefficient), but does not include modulational instability.
- Will the spectral evolution simulated with the correct and "modified" Zakharov equation be different (and, perhaps, the latter be closer to the statistical models??)

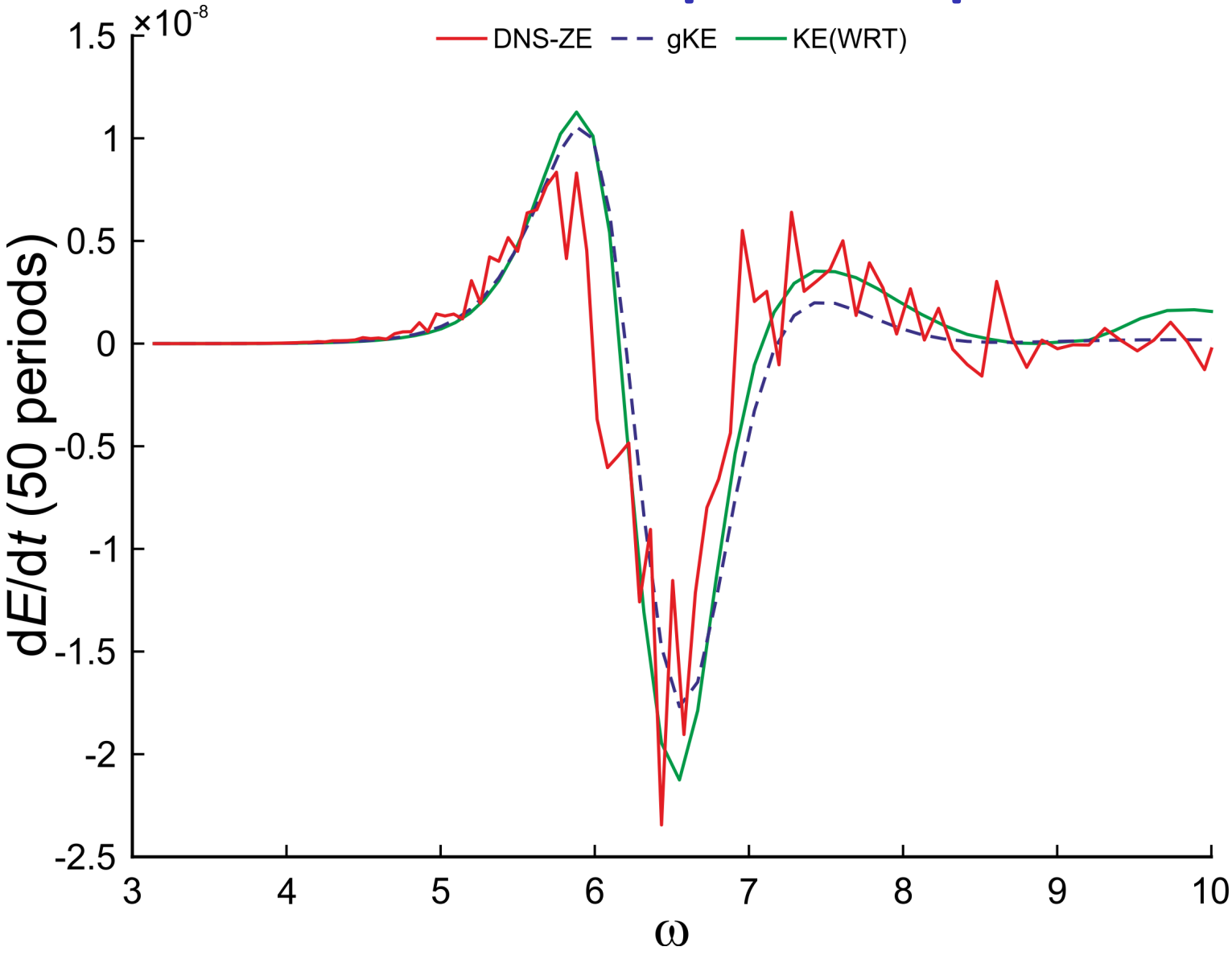
Simulations show that the hypothesis is incorrect: the spectral evolution remains the same (upon averaging over realisations).

Growth rates for different amplitudes, spectrum II



Growth rates $\mathrm{d}E(\omega, t)/\mathrm{d}t$ over first 50 periods of evolution, with DNS, KE and gKE, for amplitude multiplied by $1/\sqrt{2}$, 1 and $\sqrt{2}$

Growth rates for small amplitude, spectrum II



Growth rates $\mathrm{d}E(\omega, t)/\mathrm{d}t$ over first 50 periods of evolution, with DNS, KE and gKE, for half amplitude

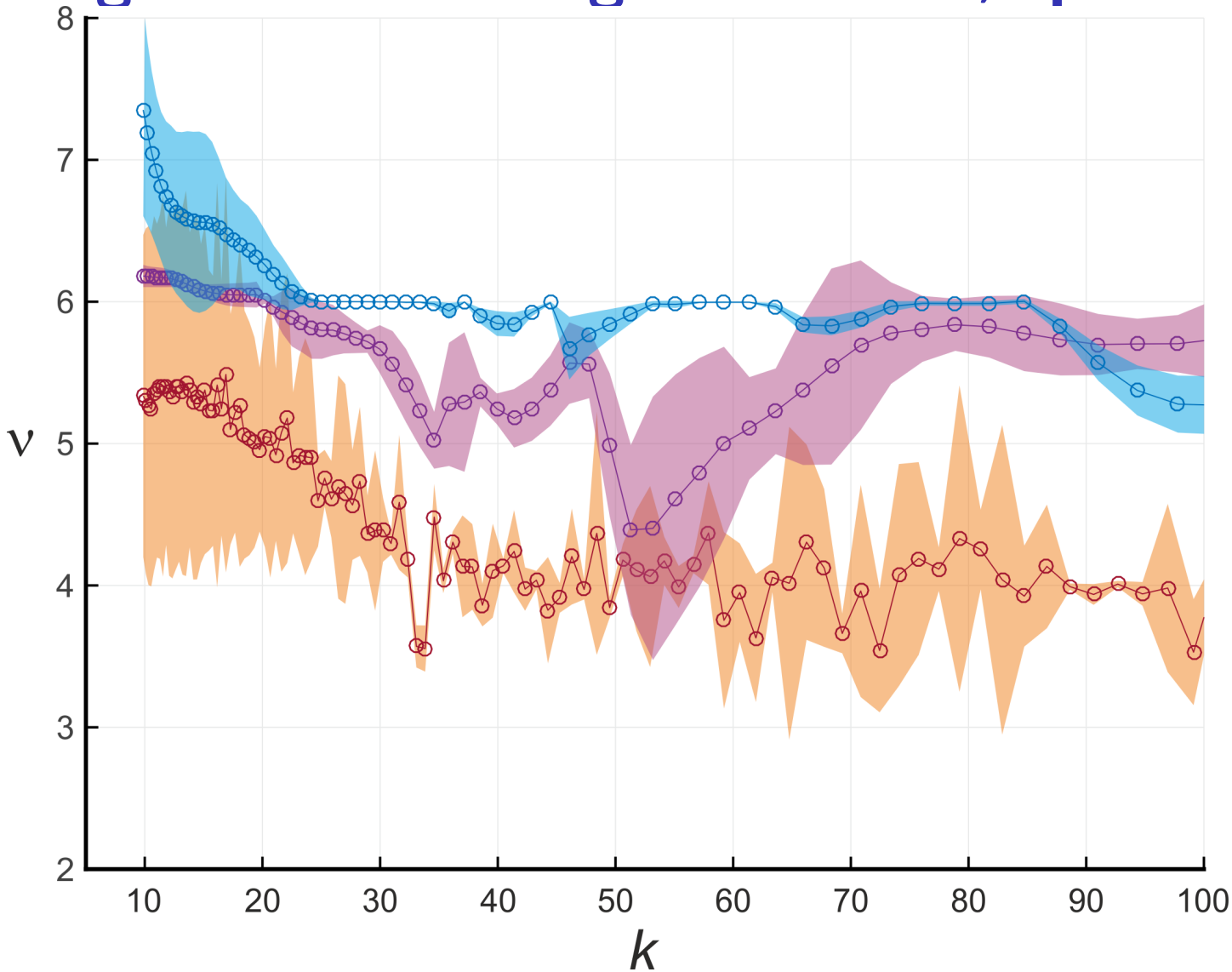
Scaling of growth rates

In order to understand how the growth rates of wave action $n(k,t)$ scale with nonlinearity within different approaches, we find the maximum value of $\mathrm{d}n/\mathrm{d}t$ and perform a numerical fit

$$\log \max \mathrm{d}n/\mathrm{d}t = \nu \log \varepsilon + \beta$$

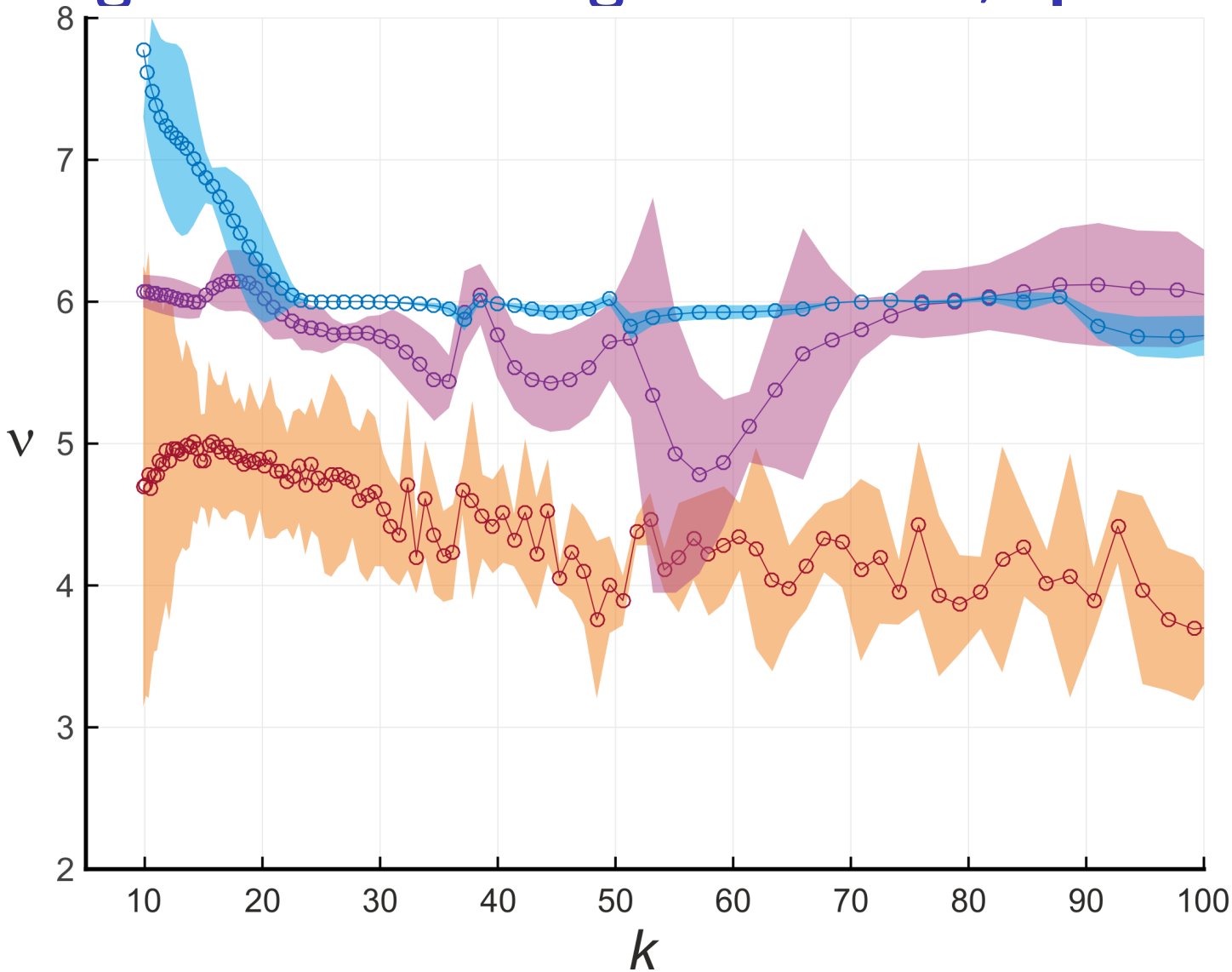
over 5 different amplitudes (different from the initial one by $1/2$, $1/\sqrt{2}$, 1, $\sqrt{2}$ and 2). Thus, we draw a straight line through 5 points by least squares, find the coefficient ν and the 95% confidence bounds for it. We know *a priori* that the KE, being an equation in real variables, has the strict $\nu = 6$ scaling (that is, $\mathrm{d}n/\mathrm{d}t \sim \varepsilon^6$). The values of ν for other approaches are to be found numerically.

Scaling for maximum growth rates, spectrum I



Exponent ν of the scaling as ε^ν for maximum growth rates for KE, gKE and DNS-ZE, and its 95% confidence bounds. The initial spectral peak is at $k = 4\pi^2 \approx 39.5$. Blue: KE(WRT), purple: gKE, orange: DNS-ZE

Scaling for maximum growth rates, spectrum II



Exponent ν of the scaling for maximum growth rates for KE, gKE and DNS-ZE, and its 95% confidence bounds. The initial spectral peak is at $k = 4\pi^2 \approx 39.5$. Blue: KE(WRT), purple: gKE, orange: DNS-ZE

Conclusions

We have considered the short- and long-term evolution of narrow spectra without wind forcing, using three different models, employing different sets of assumptions. The gKE employs the statistical closure, but is free of quasi-stationarity assumption. DNS-ZE does not depend on any statistical assumptions.

- the gKE agrees with the classic KE (WRT algorithm) for the evolution of frequency spectra where it should and diverges where it should not (for initial spectra very narrow in angle)
- gKE and DNS-ZE allow long-term simulations of spectra, which is not possible with other existing alternatives to the KE
- in the long term, all three approaches demonstrate **very close evolution of integral characteristics** of spectra, approaching for large time the theoretical asymptotes of the self-similar stage of evolution

Conclusions continued

- however, there is a striking difference for the rate of angular broadening, which is much larger for the gKE and especially for the KE, than for the DNS-ZE
- DNS-ZE results show considerably wider frequency (or wavenumber) spectra with less pronounced peak
- the DNS-ZE rates of change of the spectra scale as ε^4, whicg corresponds to the dynamical (not kinetic) timescale of evolution
- the gKE scaling of growth rates is close to the ε^6 scaling of the KE, but the exponent is distinctively less than 6
- the growth rates are close for small nonlinearity ($\varepsilon \leq 0.05$) and diverge for $\varepsilon = O(0.1)$
- the difference of growth rate scaling in the presence of self-similarity can be responsible for the difference in spectral shapes and rates of angular broadening

Discussion

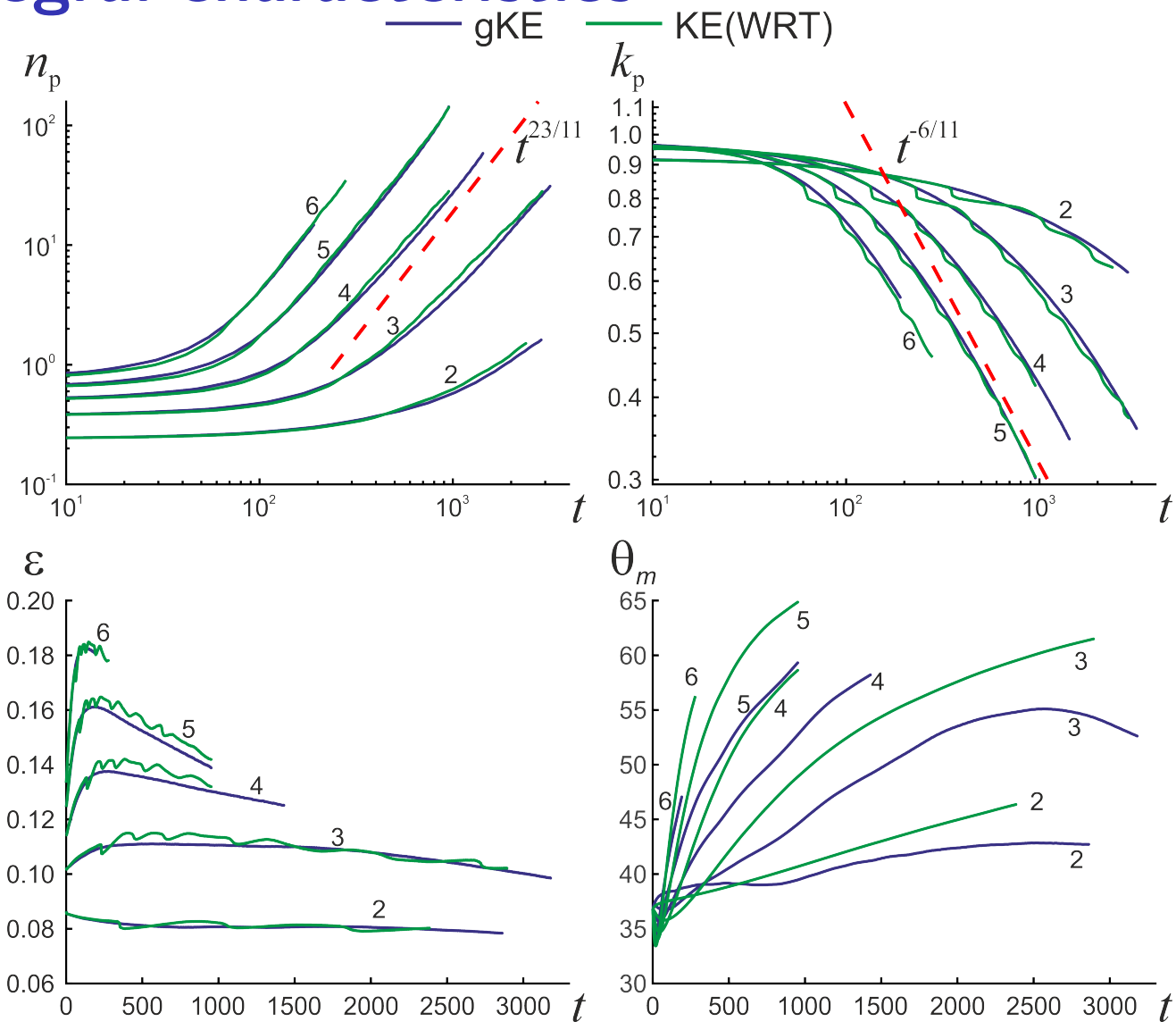
We attribute the found major discrepancies between the DNS and KE to shortcomings of the quasi-gaussian closure. It seems that at least in the maximal growth stage of field evolution an (unidentified yet) coherent component is essential.

Thank you!

simulations with constant wind

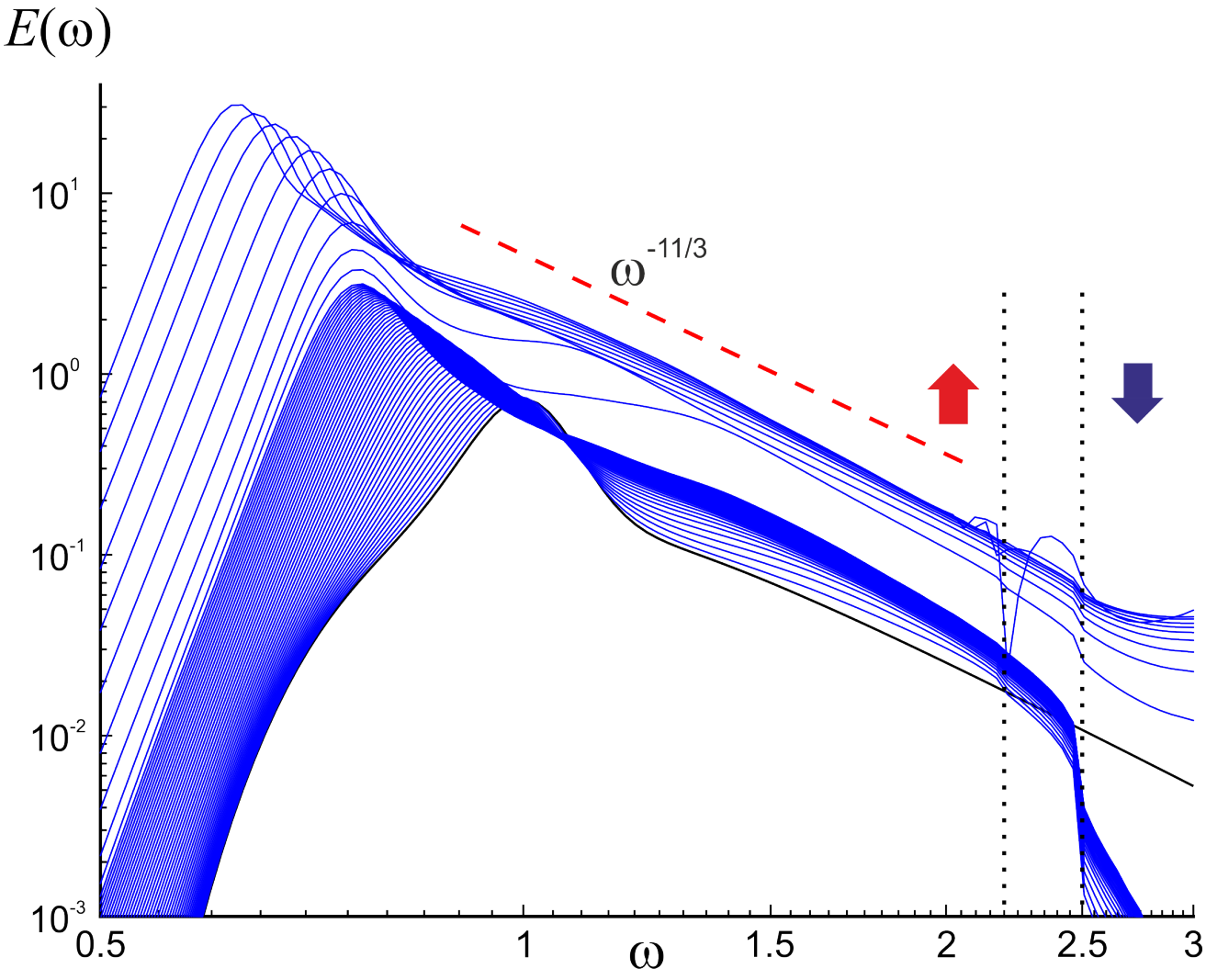
- the grid: 101 logarithmically spaced points in the range $0.5 \leq \omega \leq 3$, 31 uniformly spaced angles $-7\pi/9 \leq \theta \leq 7\pi/9$
- cutoff: $\Delta\omega/\omega_{min} \leq 0.25$
- the total number of interactions exceeds $3 \cdot 10^9$
- time stepping is performed by Runge-Kutta-Fehlberg algorithm with absolute tolerance 10^{-10} and timestep limited from above by approximately $1/3$ characteristic wave period
- initial conditions are specified as the spectra parameterised by Donelan et al (1985) for $2 \leq U_{10}/c_p \leq 6$, where c_p is the phase speed of the initial spectral peak
- initial peak frequency $\omega = 1$
- wind forcing by Hsiao and Shemdin (1983) for the corresponding U_{10}
- for comparison, the standard KE (WRT) algorithm is used, the code provided by Gerbrandt van Vledder

integral characteristics



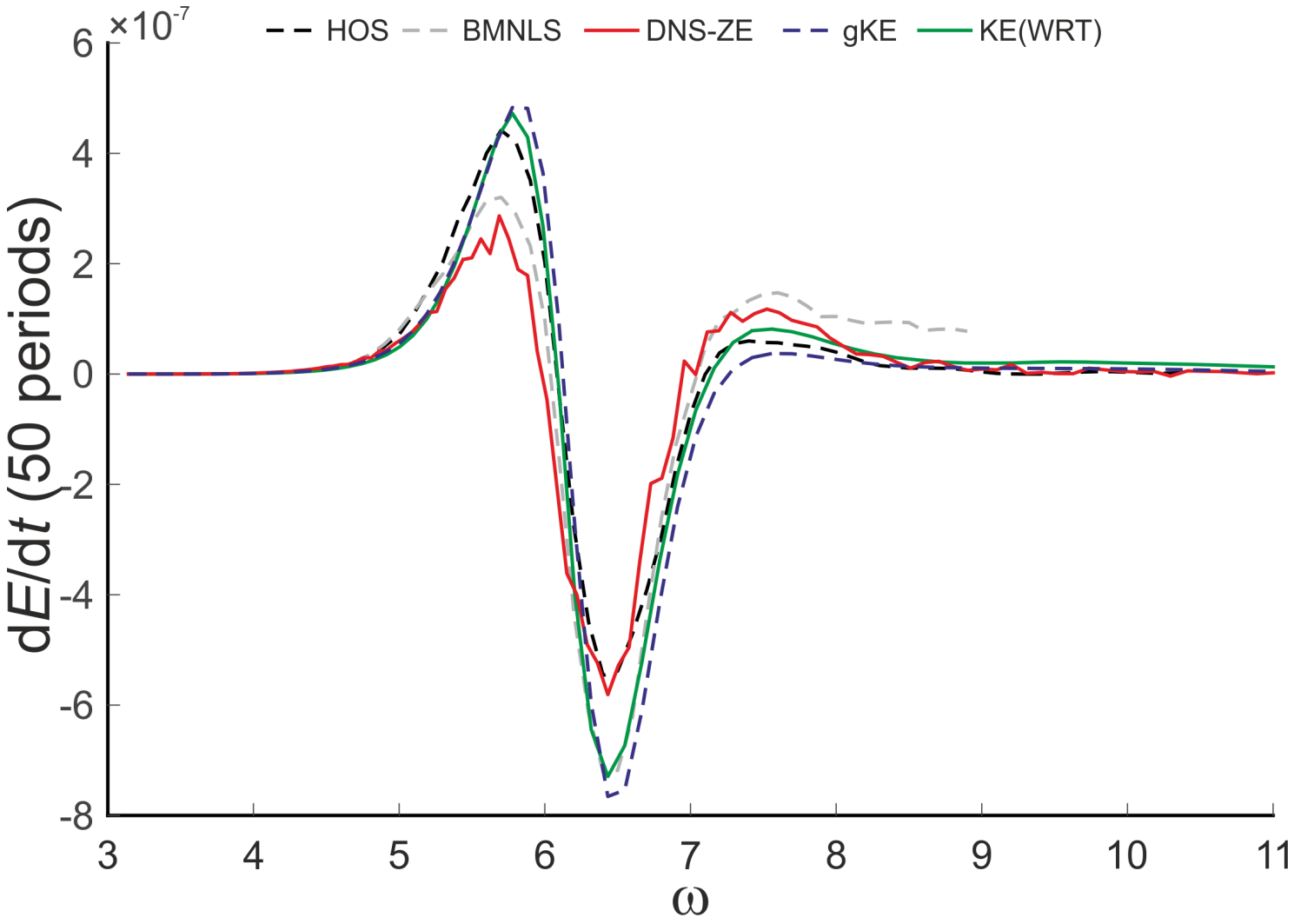
Evolution of various spectral characteristics for wind speed $2 \leq U/c_p \leq 6$. Numbers indicate the wind speed U/c_p. (a) amplitude of the peak (b) peak wavenumber (c) wave steepness (d) directional spread θ_m

instant increase of wind



Evolution of the energy spectrum $E(\omega)$ with time, under constant wind forcing with initial $U/c_p = 3$, after about 800 periods instantly increasing by a factor of 2.5. Spectra are plotted every 22 characteristic periods

growth rates over first 50 periods, spectrum II



Growth rates $\mathrm{d}E(\omega, t)/\mathrm{d}t$ over first 50 periods of evolution, with 5 approaches (values for HOS and BMNLS taken from figure 7 of Xiao et al 2013). Initial peak is at $\omega = 2\pi$