Some things I don't understand about wave turbulence

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Wave turbulence in a nutshell

Nonlinear dispersive wave equation with n-wave interaction

$$\frac{\partial a_{\mathbf{k}}}{\partial t} \sim -i\,\omega_{\mathbf{k}}\,a_{\mathbf{k}} + \int (d\mathbf{k})^{n-1}\,W^{(n)}_{\mathbf{k}}(a_{\mathbf{k}})^{n-1}\,\delta(\mathbf{k}). \qquad (1)$$

For isotropic, scale invariant systems dispersion, $\omega_{\mathbf{k}}$, and interaction coefficient, $W^{(n)}_{\mathbf{k}\,\mathbf{k}_1\ldots\mathbf{k}_{n-1}}$, can be written

$$\begin{aligned} \omega_{\mathbf{k}} &= c\,k^{\alpha} \\ W^{(n)}_{\mathbf{k}\,\mathbf{k}_1\ldots\mathbf{k}_{n-1}} &= g\,k^{\gamma_n}\,f_{\mathbf{k}\,\mathbf{k}_1\ldots\mathbf{k}_{n-1}}, \end{aligned}$$

where c and g are dimensional constants, α and γ_n are scaling exponents which are independent in principle.
Kinetic equation - statistics of weakly nonlinear limit of (1):

$$\frac{\partial n_{\mathbf{k}}}{\partial t} \sim \int (d\mathbf{k})^{n-1}\,(W^{(n)}_{\mathbf{k}})^2(n_{\mathbf{k}})^{n-1}\,\delta(\mathbf{k})\,\delta(\omega_{\mathbf{k}}). \qquad (2)$$

where the wave spectrum $n_{\mathbf{k}}$ is defined by

$$\langle a_{\mathbf{k}}\,a^{*}_{\mathbf{k}'}\rangle = n_{\mathbf{k}}\,\delta(\mathbf{k}-\mathbf{k}'). \qquad (3)$$

Is there an analogue of Kolmogorov's 4/5 Law?

For 3-D Navier-Stokes turbulence, have exact relation:

$$\left\langle \left[v_l(\vec{r}, t) - v_l(\vec{0}, t) \right]^3 \right\rangle = -\frac{4}{5}\epsilon r, \quad l \ll r \ll L. \tag{4}$$

For 4-wave turbulence in the inverse cascade regime:

$$\frac{\partial n_k}{\partial t} = \frac{\partial J_N}{\partial k} = \int \prod_{i=1}^{3} d\mathbf{k}_i \left[W^{(4)}(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) \Pi_{0,1;2,3} + \mathrm{perms} \right]. \tag{5}$$

is also exact (without weak nonlinearity assumption) where

$$\Pi_{0,1;2,3} = \int \prod_{i=0}^{3} d\Omega_i \mathrm{Im} \langle a^*_{\vec{k}} a^*_{\vec{k}_1} a_{\vec{k}_2} a_{\vec{k}_3}) \rangle.$$

Scaling and Zakharov transformation gives "exact" scaling:

$$\Pi_{0,1;2,3} \sim k^{-(\gamma_4 + 4d)}. \tag{6}$$

Are there models where an analogous statement can be made in physical space?

The generalised Phillips/critical balance spectrum

The formula

$$n_{\mathbf{k}} = c^u\, g^v\, J^w\, k^{-x}.$$

is dimensionally correct for any x if we choose

$$u = \frac{2\gamma_n + (n-1)d - (n-1)x}{(n-1)\alpha - \gamma_n}$$

$$v = -\frac{2\alpha + d - x}{(n-1)\alpha - \gamma_n}$$

$$w = \frac{(n-2)x + 2\alpha - 2\gamma_n - (n-2)d}{2((n-1)\alpha - \gamma_n)}.$$

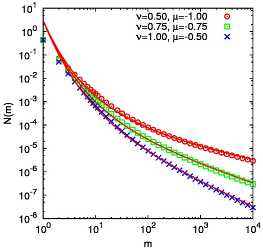
Choosing n_k independent of J ($w = 0$) gives:

$$x = \frac{2\gamma_n - 2\alpha}{n-2} + d. \tag{7}$$

This is the critical balance or generalised Phillips spectrum.
Can this scaling be related to the dynamical equations?

What are the properties of steady nonlocal cascades?

Kinetics of coagulation is equivalent to isotropic 3-wave turbulence with "backscatter" terms removed from kinetic equation. Simple enough to study nonlocality analytically.



Stationary state (theory vs numerics).

- Stationary state has the asymptotic form for $M \gg 1$:

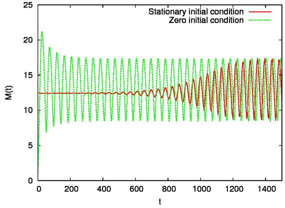
$$N_m = \frac{\sqrt{2J\gamma \log M}}{M} M^{m^{-\gamma}} m^{-\nu}.$$

$$\gamma = \nu - \mu - 1.$$

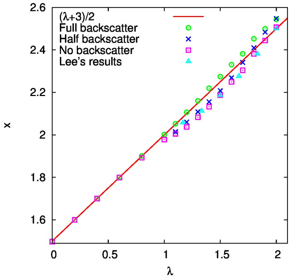
- Stretched exponential for small m, power law for large m.
- Agrees well with numerics without any adjustable parameters.

Amplitude **vanishes** as $M \to \infty$. What happens to the flux?

Time-dependent solutions of the kinetic equation?



Total mass, $M(t)$, vs time for coagulation with $\nu = 3/2$, $\mu = -3/2$.



Dynamical scaling exponents

- Instability of steady state coagulation in nonlocal regime: steady mass flux replaced by periodic pulses. Can a similar phenomenon occur for wave turbulence?
- Finite capacity anomaly: why do some finite capacity systems exhibit anomalous dynamical scaling and others don't?
- Instantaneous singularities: is the kinetic equation well posed for all physical choices of interaction coefficient?

Questions (or answers)?