Non-equilibrium condensation in WT & GP models

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BEC turbulence.

BEC is described by Gross-Pitaevskii equation:

$$i\,\frac{\partial\psi}{\partial t} + \nabla^2\psi - |\psi|^2\psi = 0. \tag{1}$$

where ψ is a complex scalar field.
GP equation (1) conserves two quantities with positive quadratic parts—the energy and the total number of particles,

$$N = \int |\psi(\mathbf{x},t)|^2 d\mathbf{x}\,, \tag{2}$$

and the total energy,

$$H = \int \left[|\nabla\psi(\mathbf{x},t)|^2 + \frac{1}{2}|\psi(\mathbf{x},t)|^4 \right] d\mathbf{x}\,, \tag{3}$$

Weak wave turbulence

Weak wave turbulence (WWT) refers to systems with random weakly nonlinear waves. In WWT, waveaction spectrum $n_{\mathbf{k}} = (L/2\pi)^d \langle |\psi_{\mathbf{k}}|^2 \rangle$ evolves according to the wave-kinetic equation (WKE):

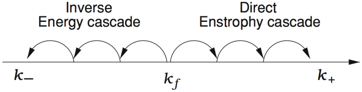
$$\partial_t n_{\mathbf{k}} = 4\pi \int |n_{\mathbf{k}_1} n_{\mathbf{k}_2} n_{\mathbf{k}_3} n_{\mathbf{k}} \left[\frac{1}{n_{\mathbf{k}}} + \frac{1}{n_{\mathbf{k}_3}} - \frac{1}{n_{\mathbf{k}_1}} - \frac{1}{n_{\mathbf{k}_2}} \right] \times$$
$$\delta(\mathbf{k} + \mathbf{k}_3 - \mathbf{k}_1 - \mathbf{k}_2)\, \delta(\omega_{\mathbf{k}} + \omega_{\mathbf{k}_3} - \omega_{\mathbf{k}_1} - \omega_{\mathbf{k}_2})\, d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3, \qquad (4)$$

where $\omega_{\mathbf{k}} = k^2$.
Now the invariants are: $N = \int n_{\mathbf{k}} d\mathbf{k}$ and $E = \int k^2 n_{\mathbf{k}} d\mathbf{k}$.

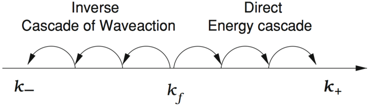
Such wave fields contain a lot of vortices but they are all ghosts!

Dual cascades

2D turbulence



Weak wave turbulence



Standard (Fjortoft'1953) argument in 2D turbulence predicts a dual cascade behaviour: energy cascades to low wavenumbers while enstrophy cascades to high wavenumbers. Similar argument in WT predicts a forward cascade of energy and an inverse cascade of waveaction (particles in the GP model.

Dual cascades in BEC

Yu. Lvov et al. / Physica D 184 (2003) 333–351

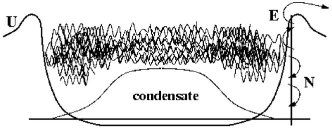
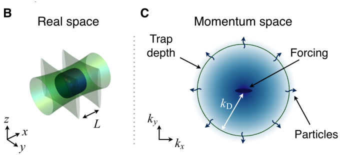


Fig. 1. Turbulent cascades of energy E and particle number N.

Navon et al.'2018.



- Direct E-cascade: "evaporation".
- Inverse N-cascade: Non-equilibrium condensation.

Kolmogorov-Zakharov spectra in the GP model

Stationary Kolmogorov-Zakharov (KZ) spectra $n_k \sim k^\nu$ are solutions of WKE corresponding to the energy and the particle cascades:

$$\nu_E = -d, \quad \curvearrowright\curvearrowright\curvearrowright$$

and

$$\nu_N = -d + 2/3, \quad \curvearrowleft\curvearrowleft\curvearrowleft\,.$$

KZ spectra are only meaningful if they are *local*, i.e. when the collision integral in the original kinetic equation converges.
In 3D ($d = 3$) the inverse $\mathcal{N}$-cascade spectrum is local, whereas the the direct E-cascade spectrum is log-divergent at the infrared (IR) limit (i.e. at $k \to 0$). As usual, the log-divergence can be remedied by a log-correction,

$$n_k \sim [\ln(k/k_f)]^{-1/3}\, k^{\nu_E},$$

where k_f is an IR cutoff provided by the forcing scale.

KZ solutions of the GP system cont'd

The 2D case ($d = 2$) appears to be even more tricky. It turns out that formally the $\mathcal{N}$-cascade spectrum is local, but the $\mathcal{N}$-flux appears to be positive, in contradiction with the Fjørtoft's argument. Further, for the E-cascade spectrum, the exponent ν_E coincides with the one of the thermodynamic E-equipartition spectrum.
As a results, the KZ spectra are not realisable in the 2D GP turbulence. Instead, "warm cascade" states are observed where the E and N k-space fluxes are on background of a thermalised background.

Direct and inverse cascades in 2d GPE

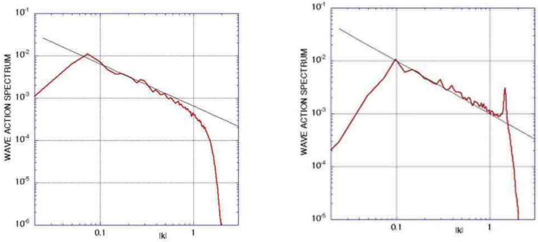
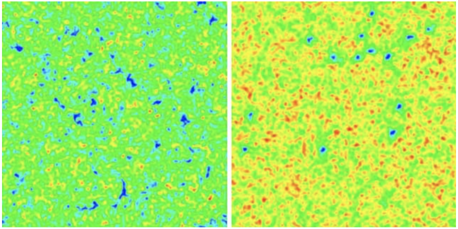


Figure: SN & M. Onorato (2006)

Both direct and inverse cascades are “warm”: their spectra are thermal equipartition of energy with small corrections to accommodate E and N fluxes.

Evolving 2d GP turbulence



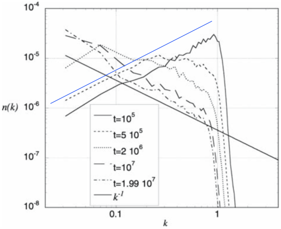


Figure: SN & M. Onorato (2007)

Evolution scenario: 4-wave WT of de-Broglie waves $\rightarrow$ hydrodynamics of point vortices $\rightarrow$ 3-wave WT of Bogoliubov sound

Direct and inverse cascades in 3d GPE

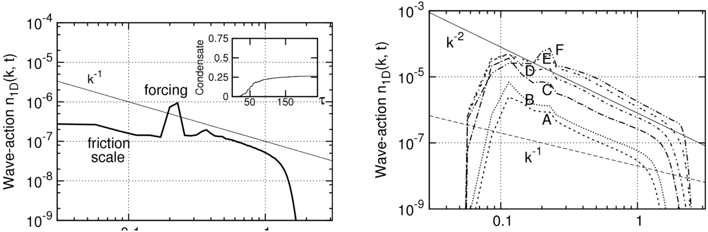


Figure: Proment, SN & M. Onorato (2012)

The spectrum is very sensitive to the type of IR dissipation: KZ for friction and Critical Balance for hypo-viscosity

Evolving weak 3D BEC turbulence

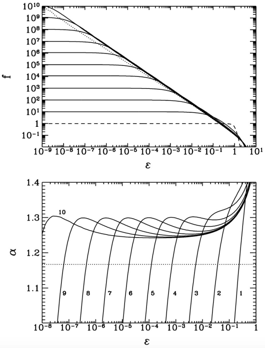
Isotropic WKE is

$$\frac{d}{dt}n_\omega = \omega^{-1/2}\int \min\left(\sqrt{\omega},\sqrt{\omega_1},\sqrt{\omega_2},\sqrt{\omega_3}\right) n_\omega n_1 n_2 n_3 \left(n_\omega^{-1}+n_1^{-1}-n_2^{-1}-n_3^{-1}\right)\delta(\omega+\omega_1-\omega_2-\omega_3)d\omega_1 d\omega_2 d\omega_3. \quad (5)$$

where $\omega = k^2$ is the wave frequency and $n_\omega(t) \sim \langle|\psi_k|^2\rangle$ is the spectrum.

Self-similar evolution in the inverse cascade range

Non-equilibrium condensation process.
(Semikoz and Tkachev 1995, Lacaze et al 2001)



Solution "blows up" in finite time t^*. Shortly before t^* they reported $n = \omega^{-x^*}$
$x^* = 1.23 > 1.16 = x_{KZ}$. Thermodynamic $n = 1/\omega$ is observed after t^*.

Self-similar formulation in the inverse cascade range

Boris Semisalov, Vladimir Grebenev, Sergey Medvedev and SN are currently working on finding the self-similar solution of WKE.
Let us search the solution of the WKE in a similarity form $n_\omega = \tau^a f(\eta)$, where $\eta = \omega\tau^{-b}, b = a - 1/2 > 0, \tau = t^* - t$. If we denote $x = \frac{a}{b}$, WKE can be rewritten as

$$xf + \eta f' = \frac{1}{b} St[f] \tag{6}$$

Self-similarity of the second type (Zeldovich): a and b cannot be found from a conservation law (e.g. energy), but are solutions of a nonlinear eigenvalue problem.
Boundary conditions:
(1) $f(\eta) \to \eta^x$ for $\eta \to \infty$.
(2) $f(\eta) \to ?$ for $\eta \to 0$.

Nonlocal interaction in the low-frequency range

For power spectrum solution $n_\omega = \omega^{-x}$, the collision integral integral converges in the range $1 < x < 3/2$.
Simulations of Semikoz and Tkachev indicate $x \approx 0$ at low ω. For such spectra the integral is divergent at infinity. Thus, the leading contribution comes from non-local interactions with $\omega_{1,2,3} \gg \omega$, so the WKE becomes

$$\frac{d}{dt} n_\omega = \int n_1 n_2 n_3 \delta(\omega_1 - \omega_2 - \omega_3) d\omega_1 d\omega_2 d\omega_3 + n_\omega \int n_1 n_2 n_3 \left(n_1^{-1} - n_2^{-1} - n_3^{-1}\right) \delta(\omega_1 - \omega_2 - \omega_3) d\omega_1 d\omega_2 d\omega_3, \quad (7)$$

where the integrals in RHS are independent of ω and n_ω. Denoting the first integral by $A(t)$ and the second – by $B(t)$, we can write (7) as

$$\frac{d}{dt} n_\omega = A(t) + B(t) n_\omega, \quad (8)$$

which can be easily integrated for any $A(t)$ and $B(t)$.

Self-similar solution for small η

For the similarity form $n_\omega = \tau^a f(\eta)$, equation (8) can be rewritten as

$$xf + \eta f' = \frac{\tilde{A}}{b} + \frac{\tilde{B}}{b} f \qquad (9)$$

where $\tilde{A}$ and $\tilde{B}$ present similarity counterpart of the integrals $A(t)$ and $B(t)$. Equation (9) can be easily integrated:

$$f(\eta) = \frac{\tilde{A}}{b(x - \tilde{B}/b)} + C\eta^{(\tilde{B}/b)-x}. \qquad (10)$$

Taking into account that $\tilde{A} \geq 0$ for $f \geq 0$ and $b > 0$ for $1 < x < 3/2$, it is easy to see that in the vicinity of $\eta = 0$ there is only one non-negative bounded solution

$$f(\eta) \to \frac{\tilde{A}}{(bx - \tilde{B})}, \qquad \eta \to 0. \qquad (11)$$

This is the second BC for the nonlinear eigenvalue problem.

Self-similar solution of WKE

$$xf + \eta f' = \frac{1}{b} St[f] \qquad (12)$$

Nonlinear eigenvalue problem: find x for which the following boundary conditions are satisfied simultaneously.

(1) $f(\eta) \to \eta^x$ for $\eta \to \infty$. (2) $f(\eta) \to$const for $\eta \to 0$.

It is much harder to solve the equation for $f(\eta)$ than to solve WKE for evolving $n(k,t)$.

Relaxation of iterations. No theory or developed numerical algorythms.

Ongoing work with B. Semisalov, V. Grebenev and S. Medvedev.

Computing the collision integral

For computation of the integrals over Δ_η we also need the values of $f(\eta)$ for $\eta > \eta_{max}$ and $\eta < \eta_{min}$. We assume that $\forall \eta > \eta_{max}$ $f(\eta) = C\eta^{-x}$ and $\forall \eta \in [0, \eta_{min}]$ $f(\eta) \equiv f(\eta_{min})$.

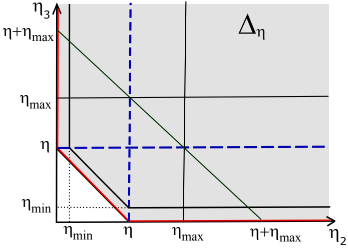
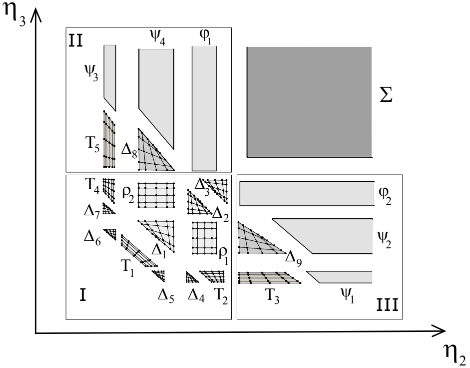


Figure: Domain of integration Δ_η (shadowed). Solid lines show the borders η_{min}, η_{max}. Dashed lines show the discontinuity of integrand's derivative due to presence of function "min". Red lines along the boundary show the singularity of integrand near zero values of η_2, η_3 and $\eta_2 + \eta_3 - \eta$

Computing the collision integral

For computation of the integrals, Δ_η was decomposed into subdomains were integrand is highly-smooth function, more precisely, Δ_η was divided into triangles, rectangles and trapezes. For high rate of convergence, we used Chebyshev approximations, since we have explicit formulas for their nodes and FFT for getting the coefficients.



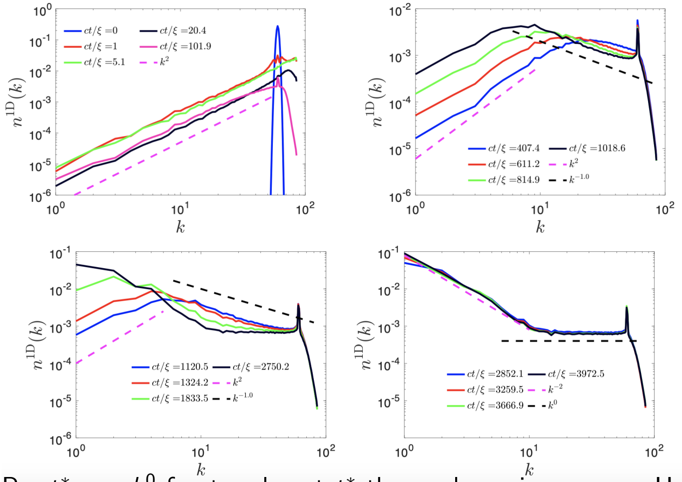
Study of the differential approximation to WKE

Ongoing work with Simon Thalabard, Sergey Medvedev, Vladimir Grebenev.

$$\partial_t n = \omega^{1-d/2} \frac{\partial^2}{\partial \omega^2} \left(\omega^s n^4 \frac{\partial^2}{\partial \omega^2} \left(\frac{1}{n} \right) \right) . \tag{13}$$

Can be transformed into a 4D autonomous dynamical system. Similar to the 2D Leith model.
Nonlinear eigenvalue problem is to find x for which the following BCs are satisfied:
(1) power law with exponent x for large frequencies.
(2) sharp front propagating to the left at which there is no forcing or dissipation.

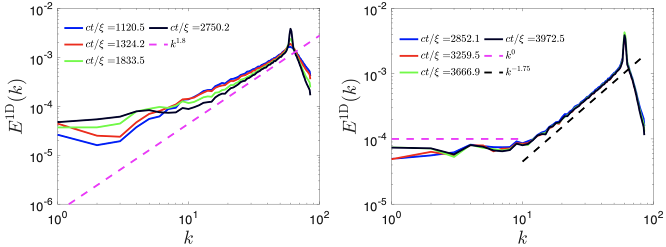
BEC turbulence. GPE DNS 256^3 of V. Shukla & SN



Pre-t^* $n = k^0$ front and post-t^* thermodynamic are seen. However, $x^* = 1$ instead of 0.46 seen for the KE solution

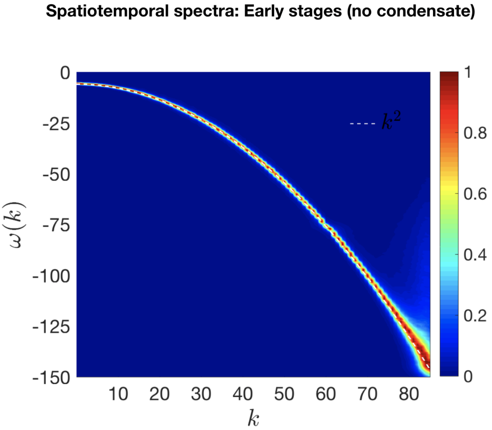
BEC turbulence. GPE DNS 256^3 of V. Shukla & SN

1D total energy spectra



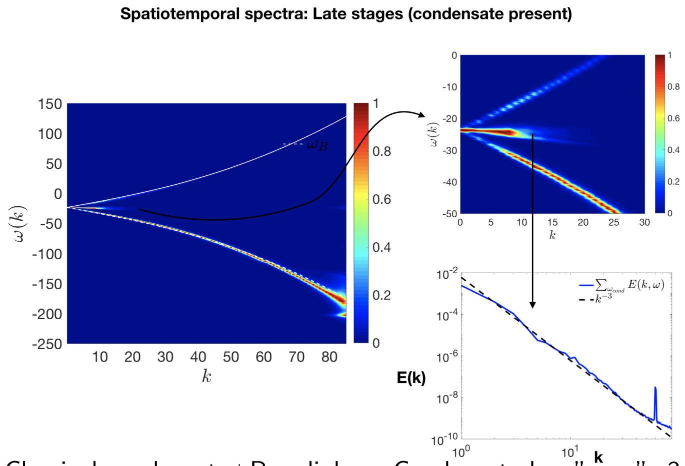
Thermal part is seen, but the condensate component is not a "delta-function" at k=0; it has a flat spectrum. Critical Balance would predict such a spectrum.

BEC turbulence. GPE DNS 256^3 of V. Shukla & SN



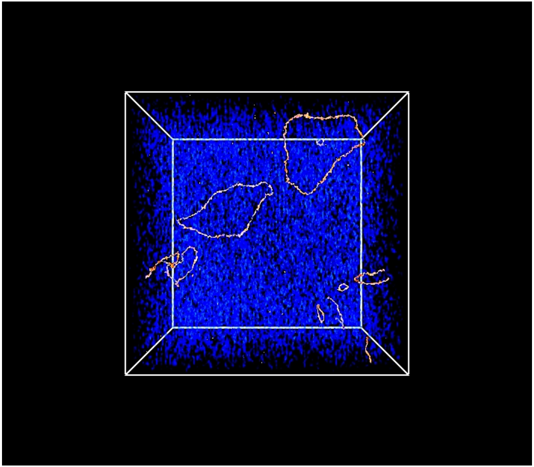
Weakly nonlinear waves, as required for the validity of the WT kinetic equations.

BEC turbulence. GPE DNS 256^3 of V. Shukla & SN



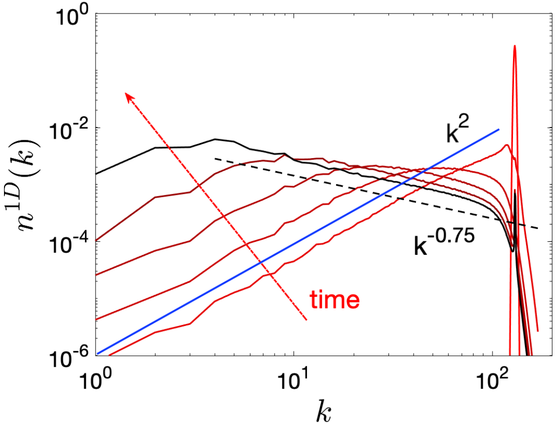
Classical condensate+Bogoliubov. Condensate has "pure" -3 scaling and it is moving!

Weak BEC turbulence. DNS 256^3 V. Shukla & SN 2018



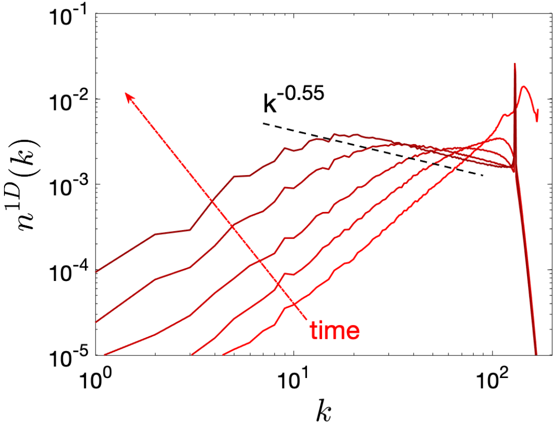
Vortices in condensate.

BEC turbulence. GPE DNS 512^3 weaker forcing



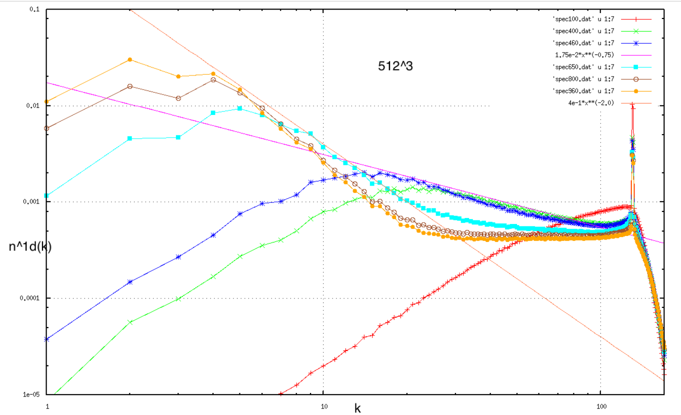
The exponent 0.75 is less than 1 now but still greater than the KE prediction of 0.46 (corresponding to $x^* = 1.23$)

BEC turbulence. GPE DNS 512^3 larger forcing



The exponent 0.55 is less than 1 now close to the KE prediction of 0.46 (corresponding to $x^* = 1.23$)

BEC turbulence. GPE DNS 512^3 of V. Shukla & SN



Condensate starts forming. High frequency spectrum gets flatter.

Summary

- Non-equilibrium condensation is characterised by a self-similar evolution with an anomalous power law scaling.
- Self-similarity of the second type: spectrum front reaches $k = 0$ at a finite time t^*.
- Post-t^* evolution is characterised by a thermal spectrum at high k and a steep power-law at low k (vortices? Critical balance?)
- Can we implement the inverse cascade KZ spectrum in laboratory by devising dissipation at low k?