Fundamentals of vortex dynamics

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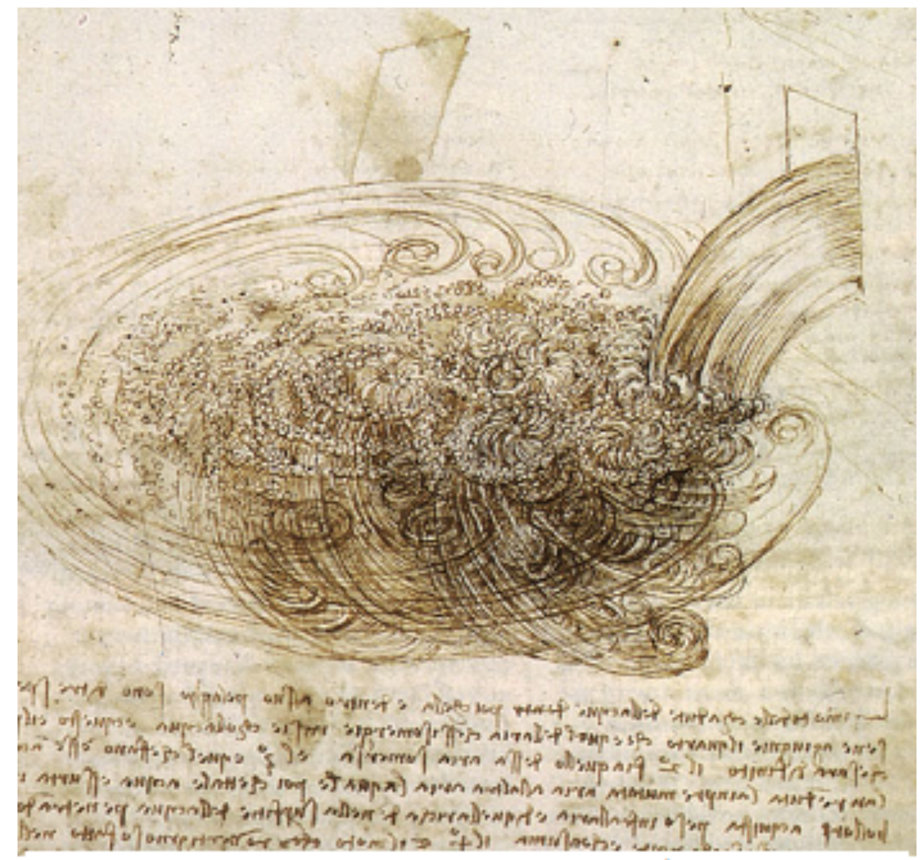
Motivation: vortices and more vortices

- wing-tip vortices (NASA)
- draining Lake Texoma, USA

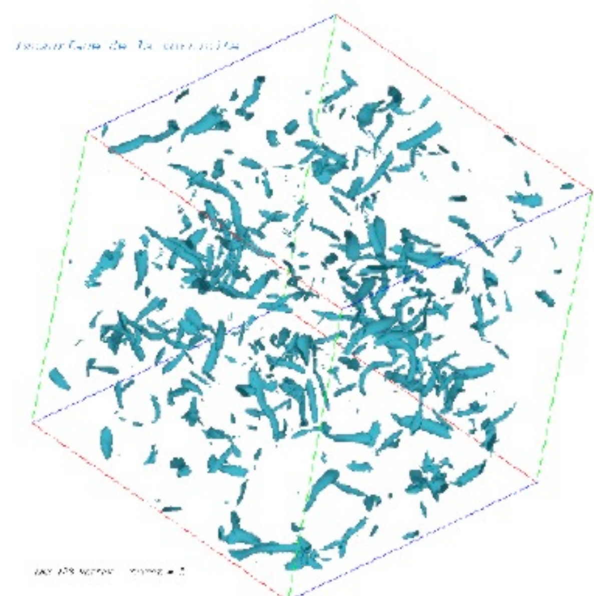
Turbulence

- Leonardo da Vinci's sketch

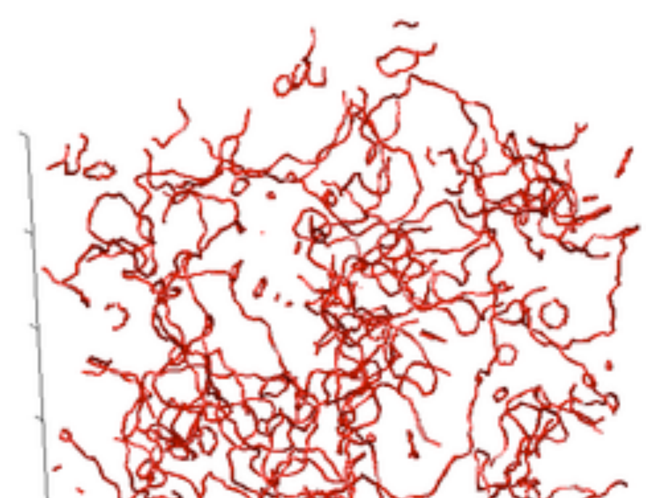


Leonardo's imagery of falling water (1508-09)
[Source: http://www.visi.com/~reuteler/leonardo.html]

- vortices in turbulence simulations

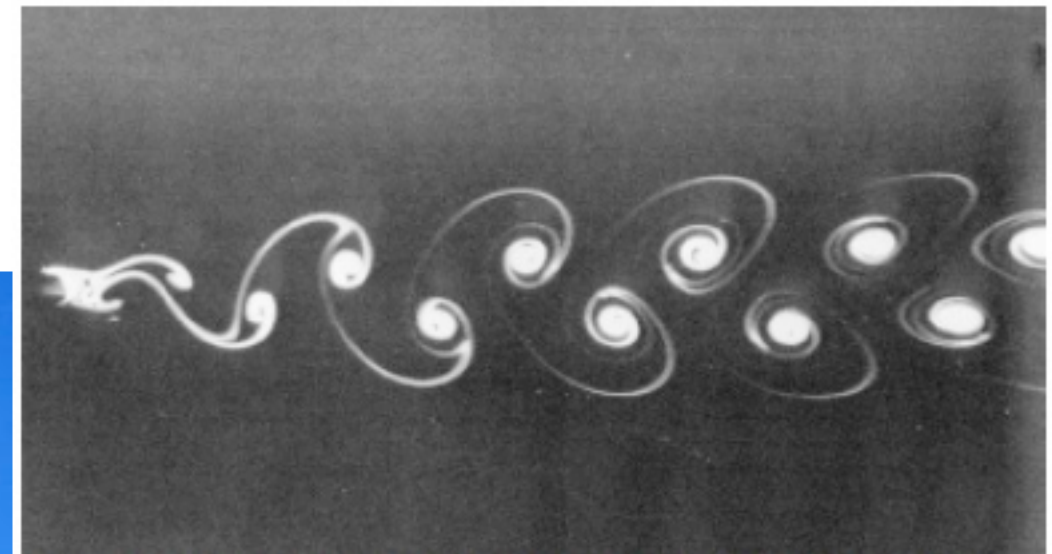


- vortices in quantum turbulence



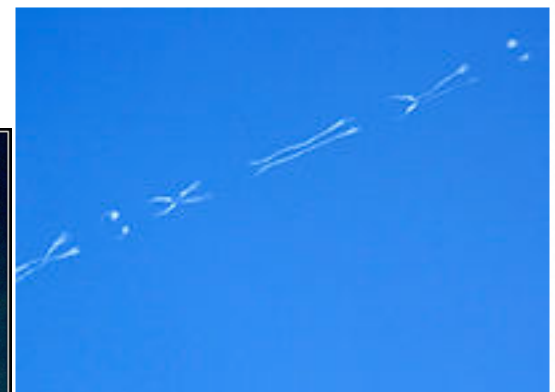
Instabilities

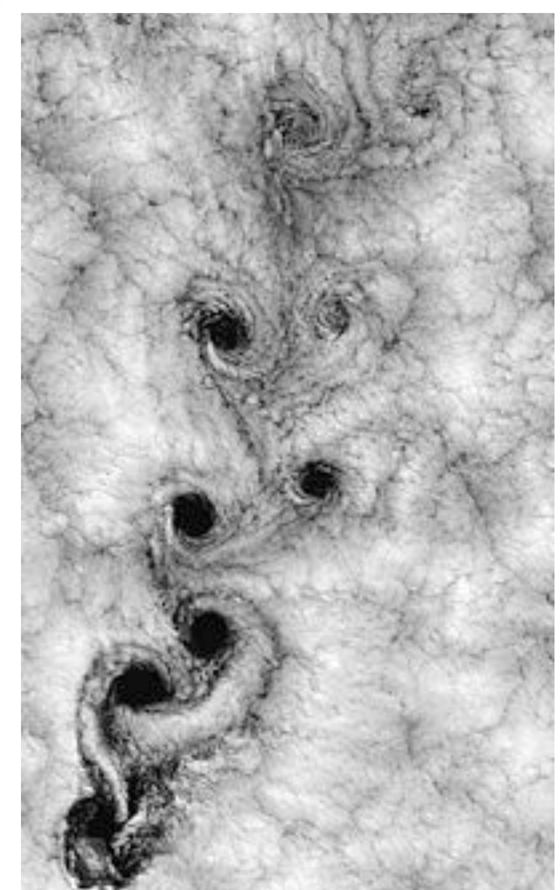
- von Karman vortex street



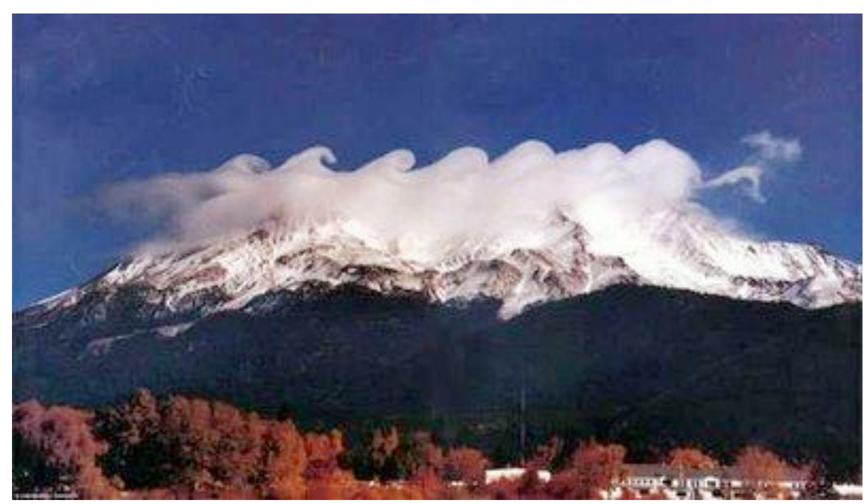
Karman vortex street behind a cylinder placed in uniform flow. Re ~ 300 [Courtesy: Sadotoshi Taneda; from An Album of Fluid Motion by Van Dyke (1982)]

- Crow instability





- Kelvin-Helmholtz instability



- Widnall vortex ring instability

Navier-Stokes and Euler equations

- Take constant density, constant viscosity, incompressible flow and write

$$D_t\boldsymbol{u} = \partial_t\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} = -\nabla p + \nu\nabla^2\boldsymbol{u}, \qquad \nabla\cdot\boldsymbol{u} = 0$$

- with kinematic viscosity $\nu = \mu/\rho$ and replacing p/ρ by p for convenience

- Euler equation for **ideal flow** $\nu = 0$ (highly singular limit)

$$D_t\boldsymbol{u} = \partial_t\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} = -\nabla p, \qquad \nabla\cdot\boldsymbol{u} = 0$$

Vorticity equation

- Take the curl of $\partial_t \boldsymbol{u} = \boldsymbol{u} \times \boldsymbol{\omega} - \nabla P$
- to obtain vorticity equation for ideal flow

$$\partial_t \boldsymbol{\omega} = \nabla \times (\boldsymbol{u} \times \boldsymbol{\omega})$$

$$D_t \omega = \partial_t \boldsymbol{\omega} + \boldsymbol{u} \cdot \nabla \boldsymbol{\omega} = \boldsymbol{\omega} \cdot \nabla \boldsymbol{u}$$

- or for non-zero viscosity

$$D_t \boldsymbol{\omega} = \partial_t \boldsymbol{\omega} + \boldsymbol{u} \cdot \nabla \omega = \boldsymbol{\omega} \cdot \nabla \boldsymbol{u} + \nu \nabla^2 \boldsymbol{\omega}$$

- eliminates pressure but still have the tricky link $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$

Vortex filament motion

- local approximations giving the motion of a thin tube of vorticity - a **vortex filament**
- by Helmholtz and Kelvin, the filament moves and stretches with the fluid motion
- we can also invert $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$ by the Biot-Savart law
- combines dynamics and differential geometry of curves

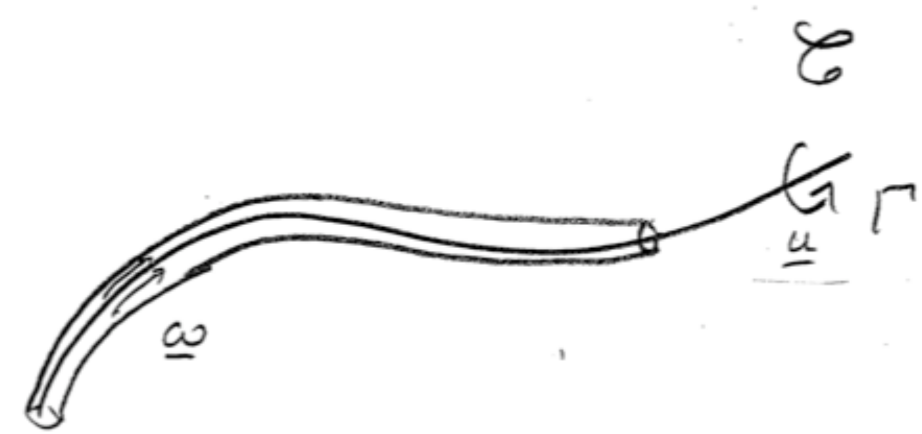
Vortex filament: Biot-Savart integral

- integral links velocity to vorticity (suppress time-dependence)

$$\boldsymbol{u}(\boldsymbol{r}) = -\frac{1}{4\pi}\int_{\mathcal{D}} \frac{(\boldsymbol{r}-\boldsymbol{r}')\times\boldsymbol{\omega}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\, dV'$$

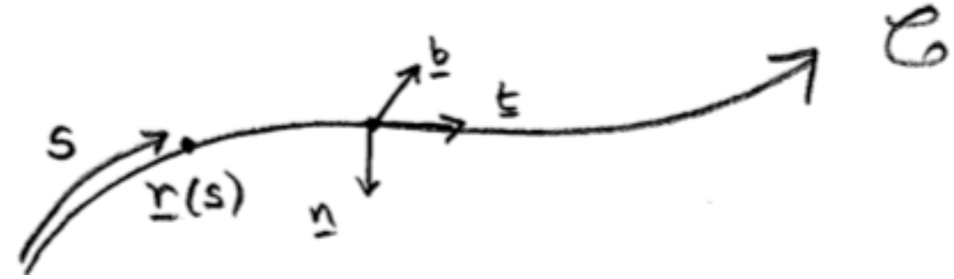
- take vorticity confined to a thin tube along a curve $\mathcal{C}$ and has circulation Γ (integral of vorticity across a surface area) of

$$\boldsymbol{u}(\boldsymbol{r}) = -\frac{\Gamma}{4\pi}\oint_{\mathcal{C}} \frac{(\boldsymbol{r}-\boldsymbol{r}')\times d\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3}$$



- orthonormal Serret-Frenet basis $\{\boldsymbol{t},\boldsymbol{n},\boldsymbol{b}\}$

$$\frac{d\boldsymbol{r}}{ds} = \boldsymbol{t}, \quad \frac{d\boldsymbol{t}}{ds} = \kappa\boldsymbol{n}, \quad \frac{d\boldsymbol{n}}{ds} = -\kappa\boldsymbol{t}+\tau\boldsymbol{b}, \quad \frac{d\boldsymbol{b}}{ds} = -\tau\boldsymbol{n}$$

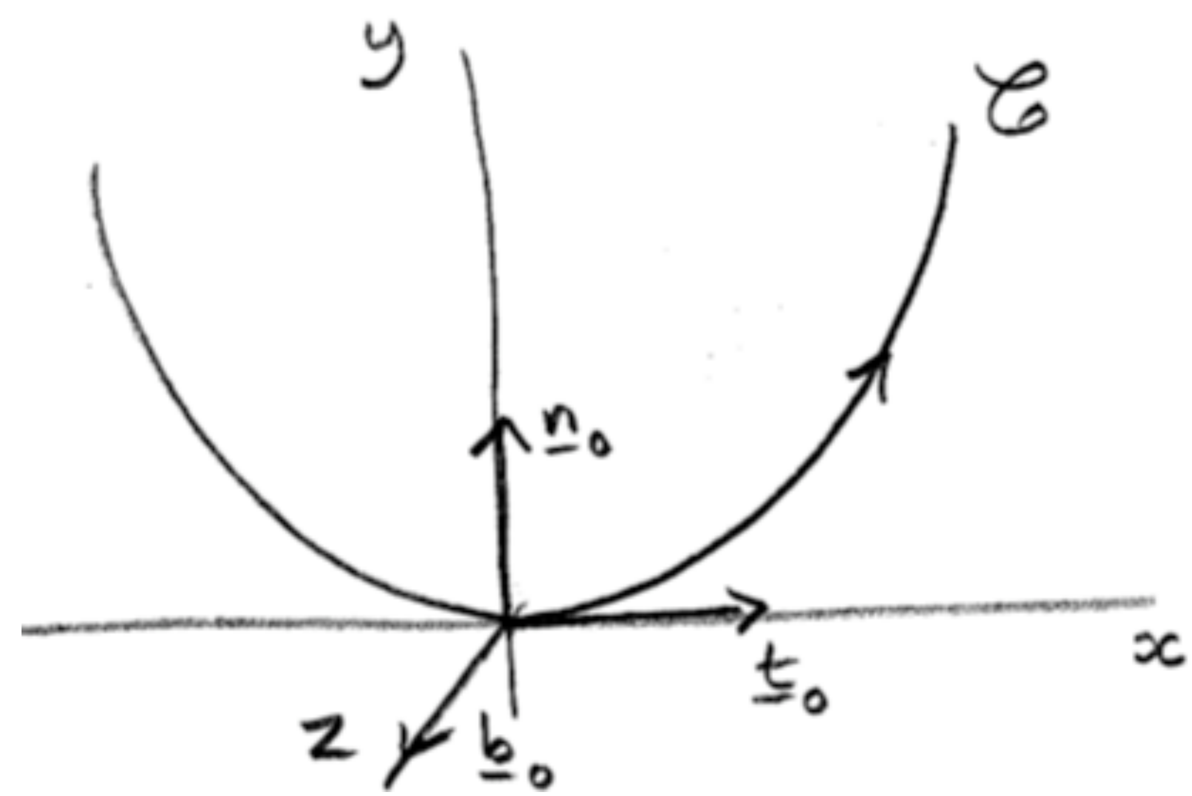


- arclength s, curvature κ, torsion τ

Local velocity from a filament

- filament through origin O with axes (x, y, z) aligned with $\{\boldsymbol{t}_0, \boldsymbol{n}_0, \boldsymbol{b}_0\}$ (at O)
- $\boldsymbol{r} = s\boldsymbol{t}_0 + \frac{1}{2}\kappa s^2 \boldsymbol{n}_0 + \cdots$ as $\boldsymbol{t} = \dfrac{d\boldsymbol{r}}{ds} = \boldsymbol{t}_0 + \kappa s \boldsymbol{n}_0 + \cdots, \quad \boldsymbol{n} = \dfrac{d\boldsymbol{t}}{ds} = \kappa \boldsymbol{n}_0 + \cdots$
- look at velocity at a point $\boldsymbol{r} = y\boldsymbol{n}_0 + z\boldsymbol{b}_0$ in plane perpendicular to vortex at O

$$\boldsymbol{u}(\boldsymbol{r}) = -\frac{\Gamma}{4\pi}\oint_C \frac{(\boldsymbol{r} - \boldsymbol{r}') \times d\boldsymbol{r}'}{|\boldsymbol{r} - \boldsymbol{r}'|^3}$$



$$d\boldsymbol{r}' = (\boldsymbol{t}_0 + \kappa s \boldsymbol{n}_0 + \cdots)\, ds$$

$$\boldsymbol{r} - \boldsymbol{r}' = -s\boldsymbol{t}_0 + (y - \tfrac{1}{2}\kappa s^2)\boldsymbol{n} + z\boldsymbol{b}_0 + \cdots$$

$$|\boldsymbol{r}' - \boldsymbol{r}|^2 = y^2 + z^2 + s^2(1 - \kappa y) + \tfrac{1}{4}\kappa^2 s^4 + \cdots$$

$$(\boldsymbol{r}' - \boldsymbol{r}) \times d\boldsymbol{r}' = [-z\kappa s \boldsymbol{t}_0 + z\boldsymbol{n}_0 - (y + \tfrac{1}{2}\kappa s^2)\boldsymbol{b}_0 + \cdots]\, ds$$

Integration to give local flow

- Biot-Savart along a filament $$\boldsymbol{u}(\boldsymbol{r}) = -\frac{\Gamma}{4\pi}\oint_C \frac{(\boldsymbol{r}-\boldsymbol{r}')\times d\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3}$$
- from a local length is $$\boldsymbol{u} = -\frac{\Gamma}{4\pi}\int_{-L}^{L} \frac{-z\kappa s\boldsymbol{t}_0 + z\boldsymbol{n}_0 - (y+\frac{1}{2}\kappa s^2)\boldsymbol{b}_0}{[y^2+z^2+s^2(1-\kappa y)+\frac{1}{4}\kappa^2 s^4]^{3/2}}\, ds$$
- put point $(0,y,z) = \sigma(0,\cos\phi,\sin\phi)$ and $\varsigma = s/\sigma$

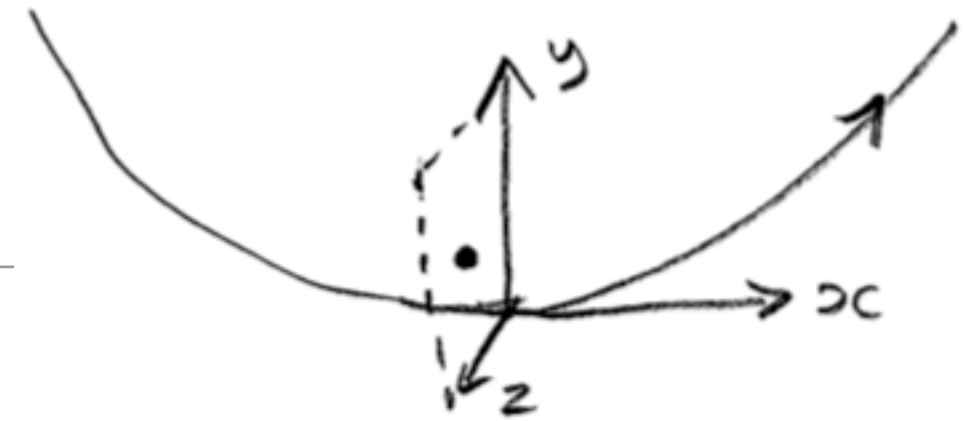
$$\boldsymbol{u} = \frac{\Gamma}{4\pi}\int_{-L/\sigma}^{L/\sigma} \frac{\sigma^{-1}(\boldsymbol{b}_0\cos\phi - \boldsymbol{n}_0\sin\phi) + \frac{1}{2}\kappa\varsigma^2\boldsymbol{b}_0}{[1+\varsigma^2(1-\kappa\sigma\cos\phi)+\frac{1}{4}\kappa^2\sigma^2\varsigma^4]^{3/2}}\, d\varsigma$$

- we want to be close to the filament $\sigma \to 0$, leaving

$$\boldsymbol{u} = \frac{\Gamma}{4\pi}\int_{-L/\sigma}^{L/\sigma} \frac{\sigma^{-1}(\boldsymbol{b}_0\cos\phi - \boldsymbol{n}_0\sin\phi) + \frac{1}{2}\kappa\varsigma^2\boldsymbol{b}_0}{(1+\varsigma^2)^{3/2}}\, d\varsigma$$

$$\boldsymbol{u} = \frac{\Gamma}{2\pi\sigma}(\boldsymbol{b}_0\cos\phi - \boldsymbol{n}_0\sin\phi) + \frac{\Gamma\kappa}{4\pi}\boldsymbol{b}_0\log\frac{L}{\sigma}$$

Local flow



- at position $(0, y, z) = \sigma(0, \cos\phi, \sin\phi)$
- flow is $\boldsymbol{u} = \dfrac{\Gamma}{2\pi\sigma}(\boldsymbol{n}_0 \sin\phi - \boldsymbol{b}_0 \cos\phi) + \dfrac{\Gamma\kappa}{4\pi}\boldsymbol{b}_0 \log\dfrac{L}{\sigma}$
- including strong local circulation, which does not move the filament
- and a weaker flow in the binormal direction $\dfrac{\Gamma\kappa}{4\pi}\boldsymbol{b}_0 \log\dfrac{L}{\sigma}$
- has a logarithmic dependence on cut-off and vortex filament width
- treat as a constant: velocity of vortex filament is now $\dfrac{\partial \boldsymbol{r}}{\partial t} = \boldsymbol{v} = C\kappa\boldsymbol{b}$
- or by rescaling time, $\dfrac{\partial \boldsymbol{r}}{\partial t} = \kappa\boldsymbol{b}$

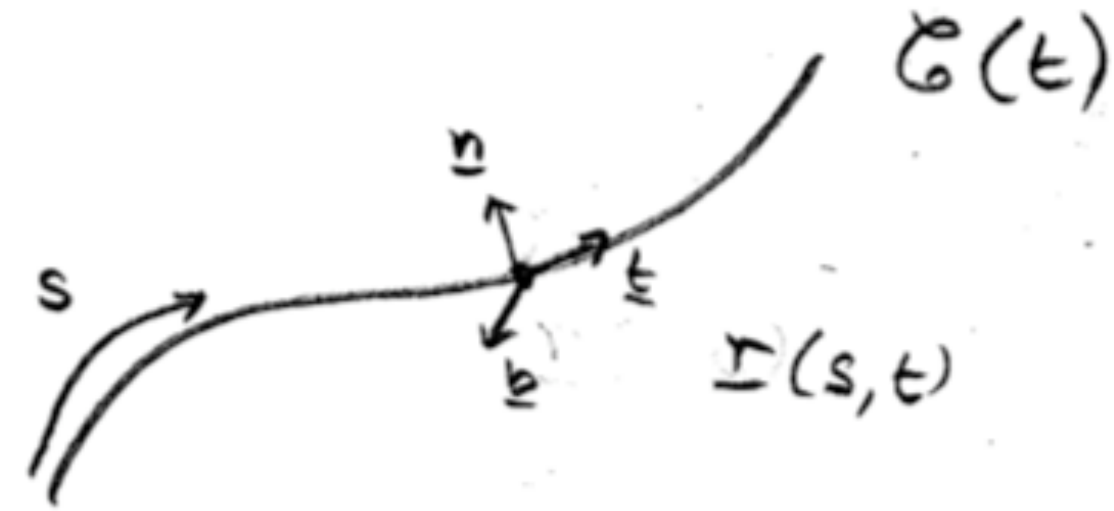
Local induction approximation (LIA)

- points $\boldsymbol{r}(s,t)$ on the curve $\mathcal{C}(t)$ with Serret-Frenet

$$\frac{\partial \boldsymbol{r}}{\partial s} = \boldsymbol{t}, \quad \frac{\partial \boldsymbol{t}}{\partial s} = \kappa \boldsymbol{n}, \quad \frac{\partial \boldsymbol{n}}{\partial s} = -\kappa \boldsymbol{t} + \tau \boldsymbol{b}, \quad \frac{\partial \boldsymbol{b}}{\partial s} = -\tau \boldsymbol{n}$$

- and velocity

$$\frac{\partial \boldsymbol{r}}{\partial t} = \kappa \boldsymbol{b}$$



- or

$$\frac{\partial \boldsymbol{r}}{\partial t} = \frac{\partial \boldsymbol{r}}{\partial s} \times \frac{\partial^2 \boldsymbol{r}}{\partial s^2}$$

- beautiful but highly idealised : no vortex stretching, only local induction, vortex width and cut-off scale fudged

Evolution of curvature and torsion - I

- dash for s derivative $\quad \boldsymbol{r}' = \boldsymbol{t}, \quad \boldsymbol{t}' = \kappa \boldsymbol{n}, \quad \boldsymbol{n}' = -\kappa \boldsymbol{t} + \tau \boldsymbol{b}, \quad \boldsymbol{b}' = -\tau \boldsymbol{n}$

- general motion (for present) $\quad \dot{\boldsymbol{r}} \equiv \boldsymbol{v} = v_t \boldsymbol{t} + v_n \boldsymbol{n} + v_b \boldsymbol{b}$

- now $\boldsymbol{t} = \boldsymbol{r}'$ and so $\dot{\boldsymbol{t}} = \dot{\boldsymbol{r}}' = \boldsymbol{v}' = A\boldsymbol{t} + B\boldsymbol{n} + C\boldsymbol{b}$ with

$$A = v_t' - \kappa v_n, \quad B = \kappa v_t + v_n' - \tau v_b, \quad C = \tau v_n + v_b'$$

- have $\dot{\boldsymbol{t}}' = \dot{\kappa} \boldsymbol{n} + \kappa \dot{\boldsymbol{n}}$ and $\dot{\boldsymbol{t}}' = A'\boldsymbol{t} + A\kappa \boldsymbol{n} + B'\boldsymbol{n} + B(-\kappa \boldsymbol{t} + \tau \boldsymbol{b}) + C'\boldsymbol{b} - C\tau \boldsymbol{n}$

- equate these gives $\dot{\boldsymbol{n}} = D\boldsymbol{t} + E\boldsymbol{n} + F\boldsymbol{b}$ with

$$\kappa D = A' - \kappa B, \quad \kappa E = \kappa A + B' - \tau C - \dot{\kappa}, \quad \kappa F = \tau B + C'$$

Evolution of curvature and torsion - II

- dash for s derivative $\boldsymbol{r}' = \boldsymbol{t}, \quad \boldsymbol{t}' = \kappa\boldsymbol{n}, \quad \boldsymbol{n}' = -\kappa\boldsymbol{t} + \tau\boldsymbol{b}, \quad \boldsymbol{b}' = -\tau\boldsymbol{n}$
- general motion (for present) $\dot{\boldsymbol{r}} \equiv \boldsymbol{v} = v_t\boldsymbol{t} + v_n\boldsymbol{n} + v_b\boldsymbol{b}$
- have $\dot{\boldsymbol{n}}' = -\dot{\kappa}\boldsymbol{t} - \kappa\dot{\boldsymbol{t}} + \dot{\tau}\boldsymbol{b} + \tau\dot{\boldsymbol{b}}$ and $\dot{\boldsymbol{n}}' = D'\boldsymbol{t} + D\kappa\boldsymbol{n} + E'\boldsymbol{n} + E(\kappa\boldsymbol{t} + \tau\boldsymbol{b}) + F'\boldsymbol{b} - F\tau\boldsymbol{n}$
- equate these gives $\dot{\boldsymbol{b}} = G\boldsymbol{t} + H\boldsymbol{n} + K\boldsymbol{b}$ with

$$\tau G = D' - \kappa(E - A) + \dot{\kappa}, \quad \tau H = \kappa(D + B) + E' - \tau F, \quad \tau K = \tau E + F' + \kappa C - \dot{\tau}$$

- have linked A, B, C, D, E, F, G, H, K to velocity components in

$$\dot{\boldsymbol{r}} \equiv \boldsymbol{v} = v_t\boldsymbol{t} + v_n\boldsymbol{n} + v_b\boldsymbol{b}$$

Evolution of curvature and torsion - III

- to close the system we use the fact that $\{\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b}\}$ is an orthonormal basis

$$\boldsymbol{t}^2 = \boldsymbol{n}^2 = \boldsymbol{b}^2 = 1, \quad \boldsymbol{t} \cdot \boldsymbol{n} = \boldsymbol{n} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{t} = 0$$

- and so $\quad 2\dot{\boldsymbol{t}} \cdot \boldsymbol{t} = 2\dot{\boldsymbol{n}} \cdot \boldsymbol{n} = 2\dot{\boldsymbol{b}} \cdot \boldsymbol{b} = 0, \quad \dot{\boldsymbol{t}} \cdot \boldsymbol{n} + \boldsymbol{t} \cdot \dot{\boldsymbol{n}} = \dot{\boldsymbol{n}} \cdot \boldsymbol{b} + \boldsymbol{n} \cdot \dot{\boldsymbol{b}} = \dot{\boldsymbol{b}} \cdot \boldsymbol{t} + \boldsymbol{b} \cdot \dot{\boldsymbol{t}} = 0$

- with

$$\dot{\boldsymbol{t}} = \dot{\boldsymbol{r}}' = \boldsymbol{v}' = A\boldsymbol{t} + B\boldsymbol{n} + C\boldsymbol{b}$$

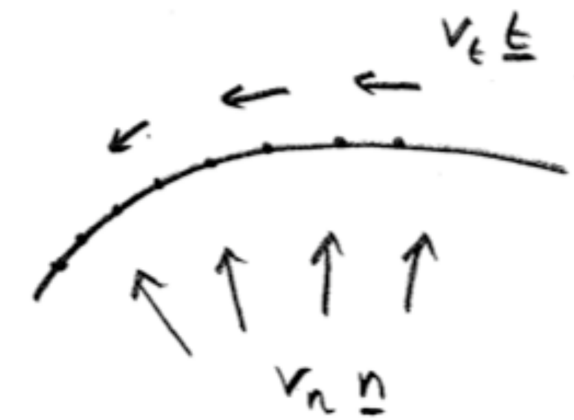
$$\dot{\boldsymbol{n}} = D\boldsymbol{t} + E\boldsymbol{n} + F\boldsymbol{b}$$

$$\dot{\boldsymbol{b}} = G\boldsymbol{t} + H\boldsymbol{n} + K\boldsymbol{b}$$

- we have $\quad A = E = K = 0, \quad D + B = 0, \quad G + C = 0, \quad H + F = 0$

Equations for curvature and torsion

- $A = 0$ gives equation from arc-length parameterisation



$$v_t' = \kappa v_n$$

- $E = 0$ and $K = 0$ give

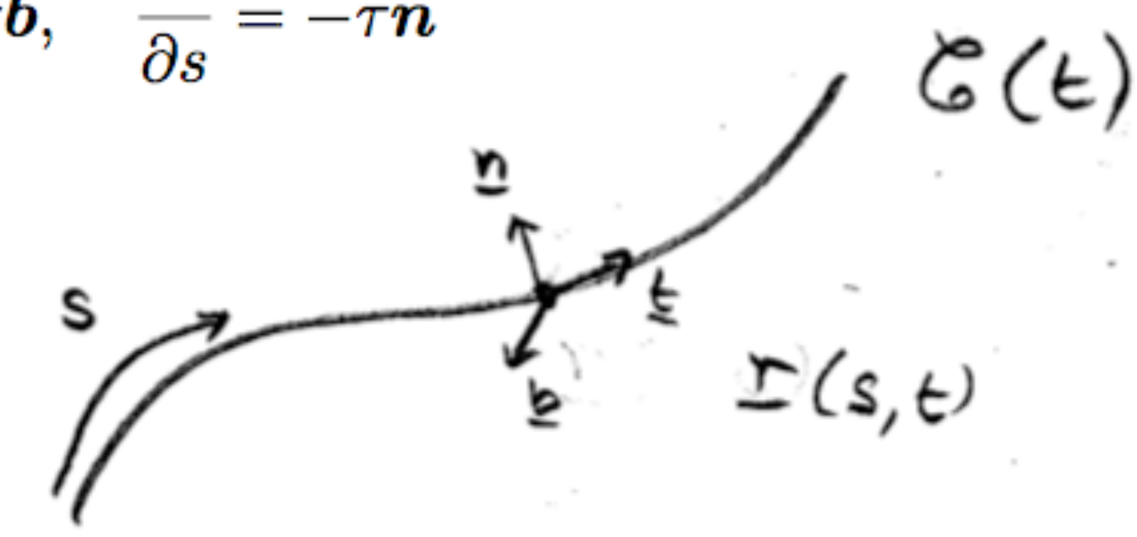
$$\dot{\kappa} = (\kappa v_t + v_n' - \tau v_b)' - (\tau v_n + v_b')\tau$$
$$\dot{\tau} = [\kappa^{-1}(\kappa v_t + v_n' - \tau v_b)\tau + \kappa^{-1}(\tau v_n + v_b')']' + (\tau v_n + v_b')\kappa$$

- or for LIA $v_n = v_t = 0 \quad v_b = \kappa$

$$\dot{\kappa} = -\kappa\tau' - 2\kappa'\tau$$
$$\dot{\tau} = (\kappa^{-1}\kappa'' - \tau^2)' + \kappa\kappa'$$

Equations for curvature and torsion under LIA

$$\frac{\partial \boldsymbol{r}}{\partial s} = \boldsymbol{t}, \quad \frac{\partial \boldsymbol{t}}{\partial s} = \kappa \boldsymbol{n}, \quad \frac{\partial \boldsymbol{n}}{\partial s} = -\kappa \boldsymbol{t} + \tau \boldsymbol{b}, \quad \frac{\partial \boldsymbol{b}}{\partial s} = -\tau \boldsymbol{n}$$



- a lot of manipulation… gives

$$\dot{\kappa} = -\kappa\tau' - 2\kappa'\tau$$

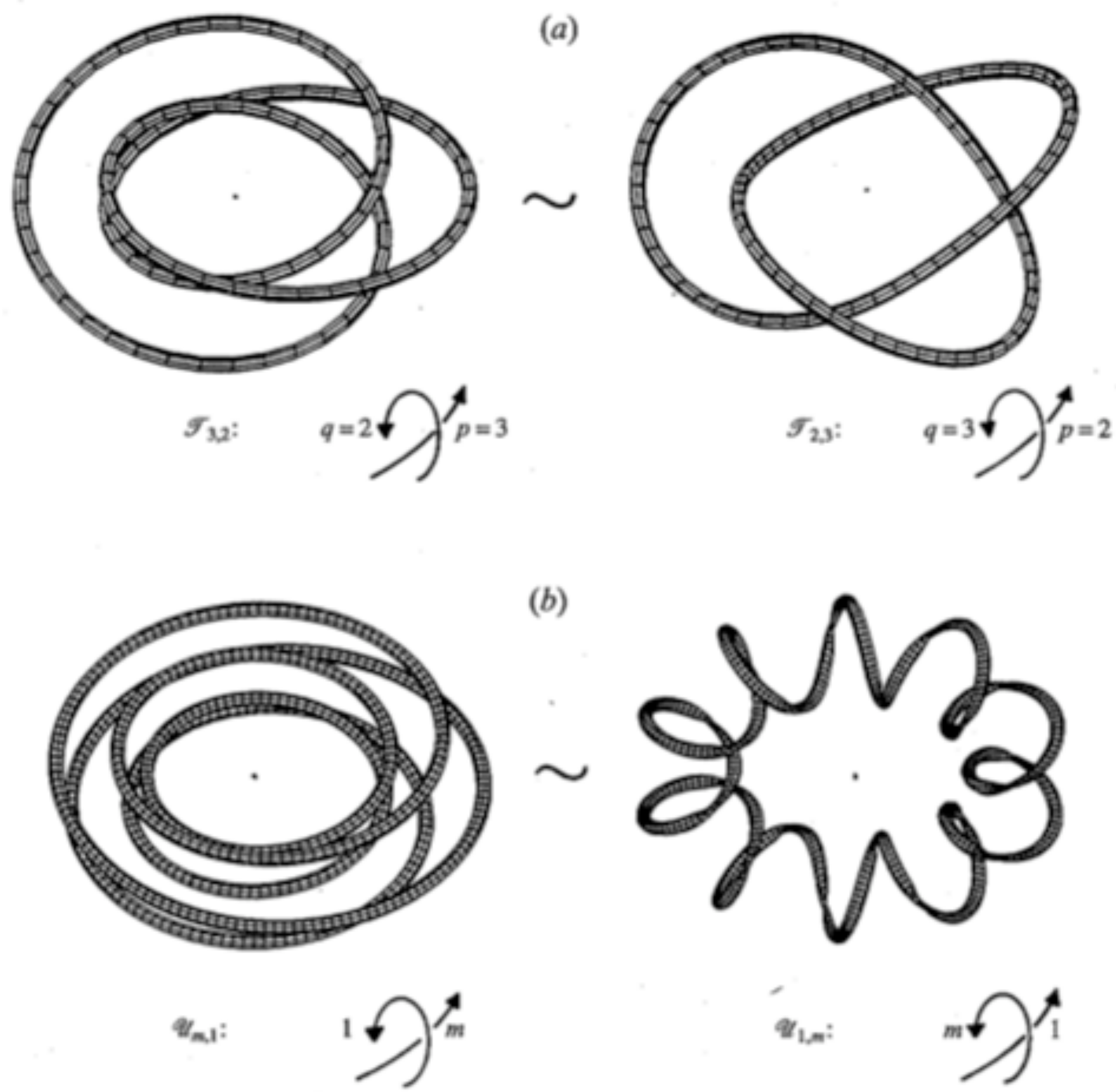
$$\dot{\tau} = (\kappa^{-1}\kappa'' - \tau^2)' + \kappa\kappa'$$

- prime denotes derivative with respect to arclength

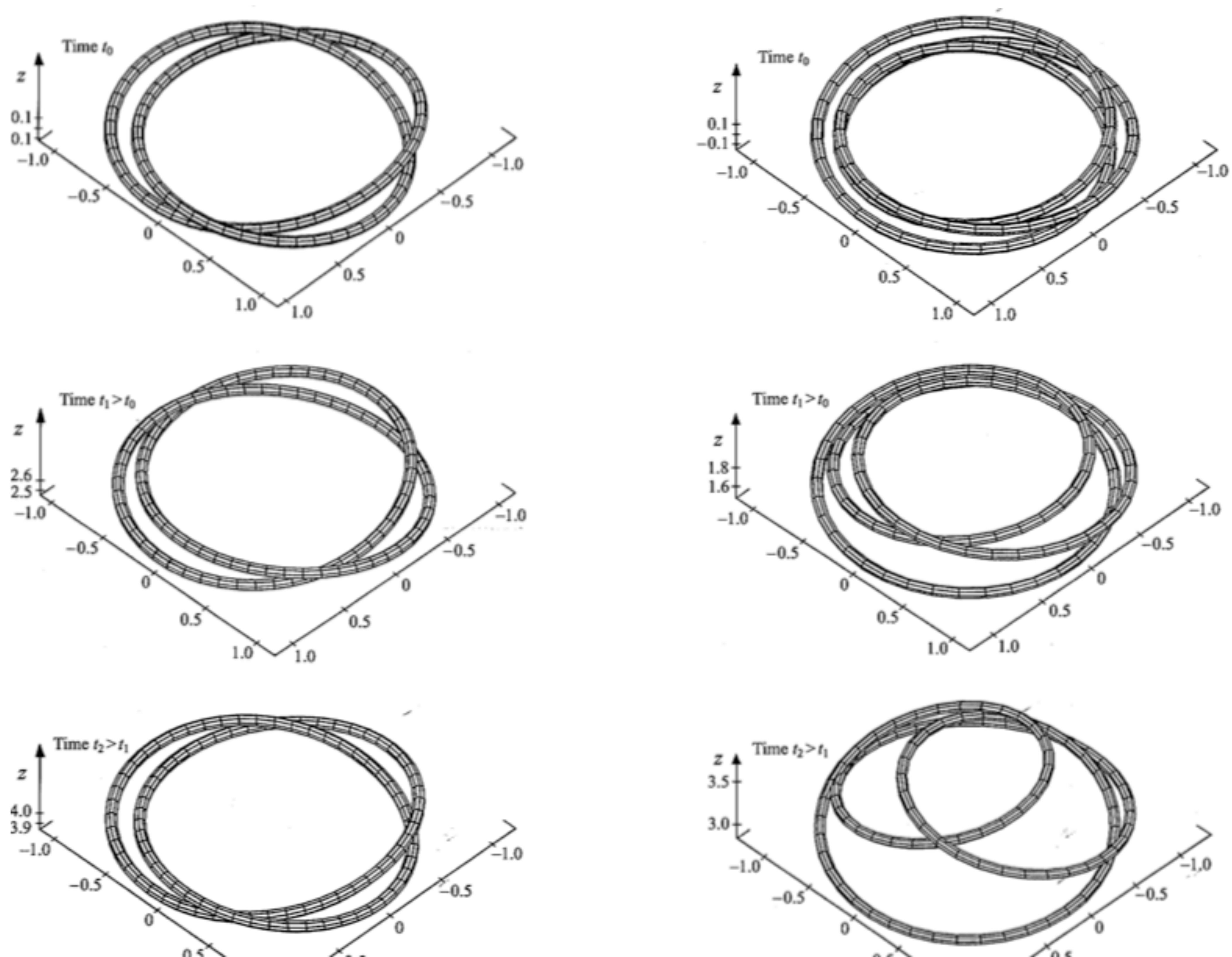
- …link to nonlinear Schrodinger equation (integrable PDE)…

Knot evolution under LIA

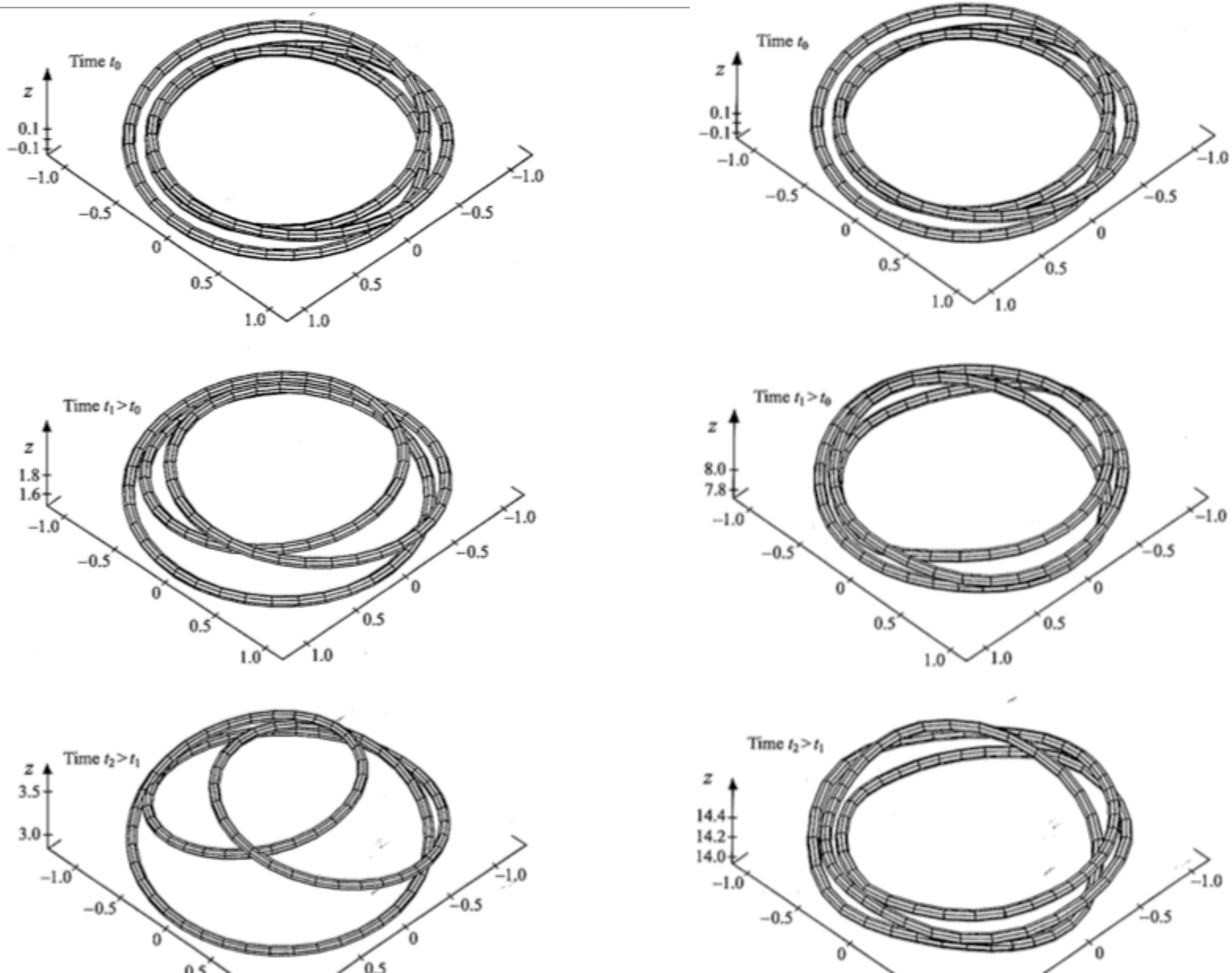
- Ricca, Samuels, Barenghi: evolve a torus knot under LIA



Evolution of F(2,3) and F(3,2) under LIA



Evolution of F(3,2) under LIA and Biot-Savart

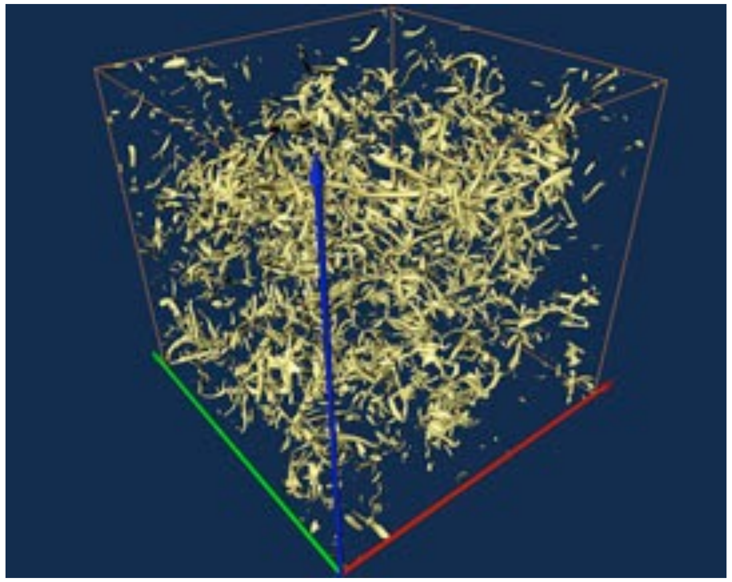


William Irvine and collaborators (Chicago)

- vortex rings created by dragging a knotted aerofoil through water:
- https://www.youtube.com/watch?v=YCA0VIExVhg (1:10)
- https://www.youtube.com/watch?v=9CnilX-oLrI
- https://www.youtube.com/watch?v=LdOX24KwSUU
- https://www.youtube.com/watch?v=CoUglS21w6c

Vortex stretching

- this important phenomenon is not in the LIA though it appears in more sophisticated models



Jörg Schumacher

- intense fine-scale vortices seen in 3-d turbulence
- vortex stretching creates fine scales
- question of the regularity of the 3-d Euler equation:
- starting with smooth initial conditions, does the solution remain smooth for all time?
- fundamental, unsolved problem:

Clay Millenium prizes

- In order to celebrate mathematics in the new millennium, The Clay Mathematics Institute of Cambridge, Massachusetts (CMI) has named seven Prize Problems. The Scientific Advisory Board of CMI selected these problems, focusing on important classic questions that have resisted solution over the years.

- Birch and Swinnerton-Dyer Conjecture
- Hodge Conjecture
- Navier-Stokes Equations
- P vs NP
- Poincaré Conjecture --- proven!
- Riemann Hypothesis
- Yang-Mills Theory

Navier-Stokes equations

$$\frac{\partial}{\partial t}u_i + \sum_{j=1}^{n} u_j \frac{\partial u_i}{\partial x_j} = \nu \Delta u_i - \frac{\partial p}{\partial x_i} + f_i(x,t) \qquad (x \in \mathbb{R}^n, t \geq 0), \qquad u(x,0) = u^\circ(x) \qquad (x \in \mathbb{R}^n).$$

$$\operatorname{div} u = \sum_{i=1}^{n} \frac{\partial u_i}{\partial x_i} = 0 \qquad (x \in \mathbb{R}^n, t \geq 0)$$

(A) Existence and smoothness of Navier–Stokes solutions on $\mathbb{R}^3$. Take $\nu > 0$ and $n = 3$. Let $u^\circ(x)$ be any smooth, divergence-free vector field satisfying (4). Take $f(x,t)$ to be identically zero. Then there exist smooth functions $p(x,t), u_i(x,t)$ on $\mathbb{R}^3 \times [0,\infty)$ that satisfy (1), (2), (3), (6), (7).

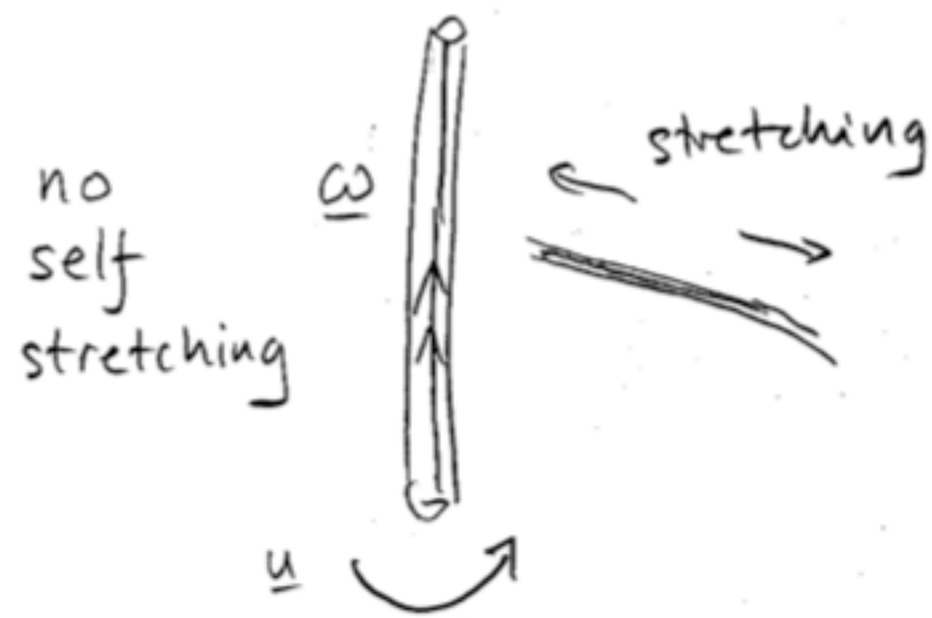
(B) Existence and smoothness of Navier–Stokes solutions in $\mathbb{R}^3/\mathbb{Z}^3$. Take $\nu > 0$ and $n = 3$. Let $u^\circ(x)$ be any smooth, divergence-free vector field satisfying (8); we take $f(x,t)$ to be identically zero. Then there exist smooth functions $p(x,t)$, $u_i(x,t)$ on $\mathbb{R}^3 \times [0,\infty)$ that satisfy (1), (2), (3), (10), (11).

(C) Breakdown of Navier–Stokes solutions on $\mathbb{R}^3$. Take $\nu > 0$ and $n = 3$. Then there exist a smooth, divergence-free vector field $u^\circ(x)$ on $\mathbb{R}^3$ and a smooth $f(x,t)$ on $\mathbb{R}^3 \times [0,\infty)$, satisfying (4), (5), for which there exist no solutions (p,u) of (1), (2), (3), (6), (7) on $\mathbb{R}^3 \times [0,\infty)$.

(D) Breakdown of Navier–Stokes Solutions on $\mathbb{R}^3/\mathbb{Z}^3$. Take $\nu > 0$ and $n = 3$. Then there exist a smooth, divergence-free vector field $u^\circ(x)$ on $\mathbb{R}^3$ and a smooth $f(x,t)$ on $\mathbb{R}^3 \times [0,\infty)$, satisfying (8), (9), for which there exist no solutions (p,u) of (1), (2), (3), (10), (11) on $\mathbb{R}^3 \times [0,\infty)$.

These problems are also open and very important for the Euler equations ($\nu = 0$), although the Euler equation is not on the Clay Institute's list of prize problems.

Idealised vorticity stretching



- full equation $$D_t\boldsymbol{\omega} = \boldsymbol{\omega}\cdot\nabla\boldsymbol{u}$$
- idealised ODE $$\frac{d\omega}{dt} = \omega^2$$
- solution $$\omega(t) = (\omega_0^{-1} - t)^{-1}, \quad \omega(0) = \omega_0$$
- singular blow-up at time ω_0^{-1}
- but: vorticity tends to stretch perpendicular vorticity, not itself
- problem of geometrical complexity
- e.g. no stretching (no singularity) in two dimensions

Beale-Kato-Majda theorem

- rigorous result
- Suppose we start with a smooth Euler flow at time $t = 0$ and that at time $t = t^*$ it is no longer smooth. Then, necessarily

$$\int_0^t \max_{\boldsymbol{r}} |\boldsymbol{\omega}(\boldsymbol{r}, s)| \, ds \to \infty \quad \text{as} \quad t \to t^*$$

- clear numerical criterion to capture any loss of smoothness
- eliminates certain types of singularities, e.g. if the maximum $\omega \sim (t^* - t)^{-\beta}$ then $\beta \geq 1$

Exact solutions of blow-up

- Let A be any symmetric trace-free matrix, then

$$\boldsymbol{u} = (t^* - t)^{-1} A\boldsymbol{r}, \quad p = -\tfrac{1}{2}(t^* - t)^{-2}\,\boldsymbol{r}\cdot(A + A^2)\cdot\boldsymbol{r}$$

- satisfies the Euler equation. But infinite energy, blows up everywhere at once, even in 2-d

- flows of the form $\boldsymbol{u} = (f(x,t), yg(x,t), zh(x,t))$

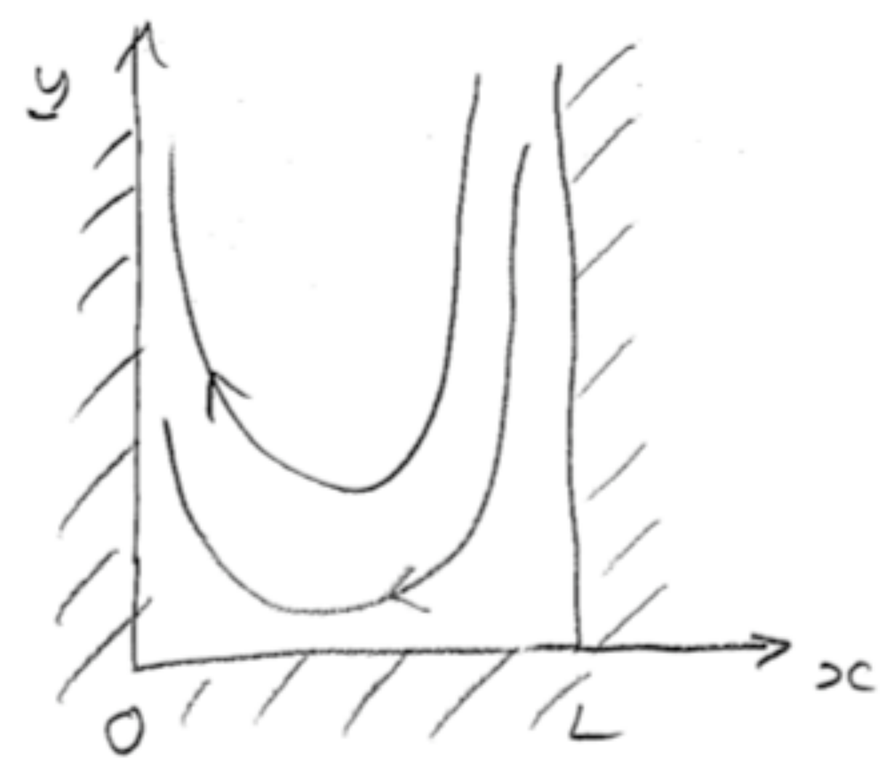
- e.g., in 2-d channel $\boldsymbol{u} = (f(x,t), -yf_x(x,t), 0)$

- can show blow-up, e.g.,

$$f(x,0) = \tfrac{1}{2}x(2-x)$$

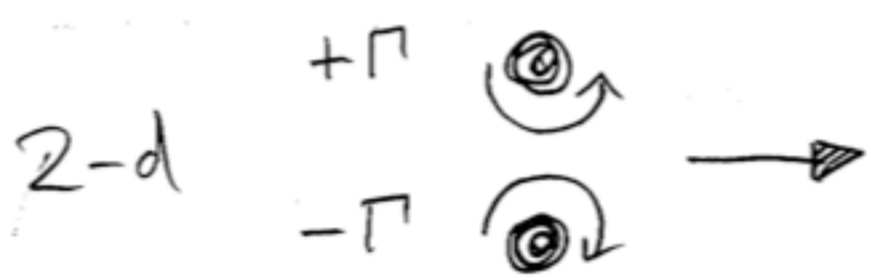
$$L = 2$$

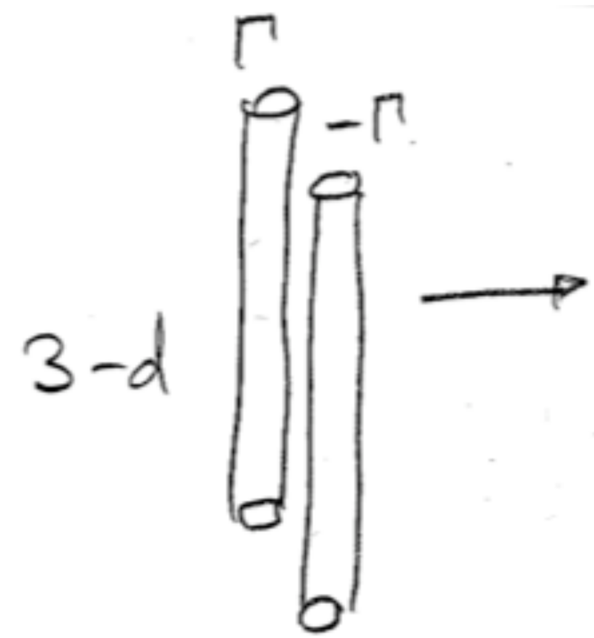
$$t^* = \pi^2/6.$$



Colliding vortices

- in 2-d a vortex pair of opposite signs translates, and similarly in 3-d



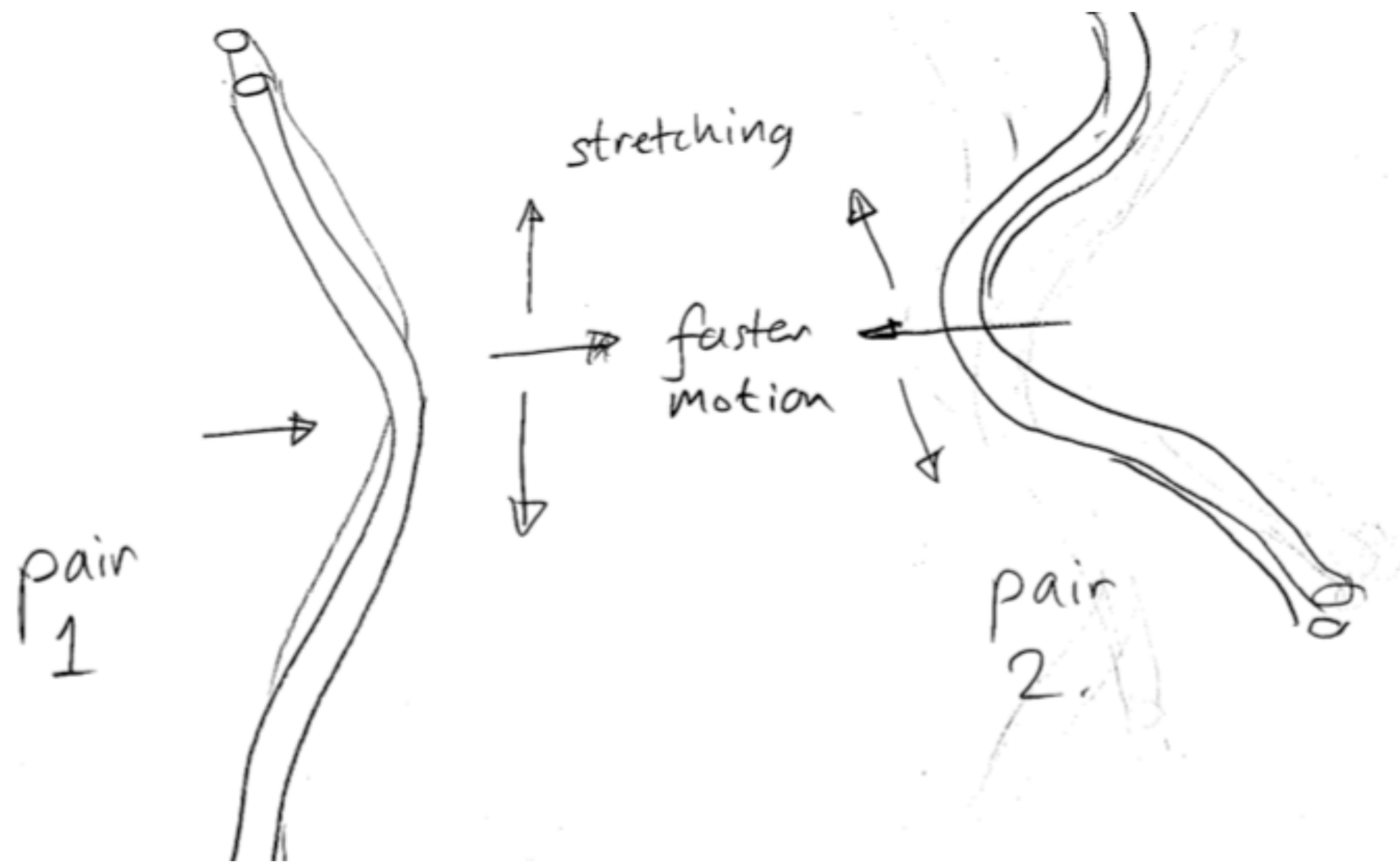


- no vortex stretching though

- try two pairs at right angles

Colliding vortex pairs: Moffatt

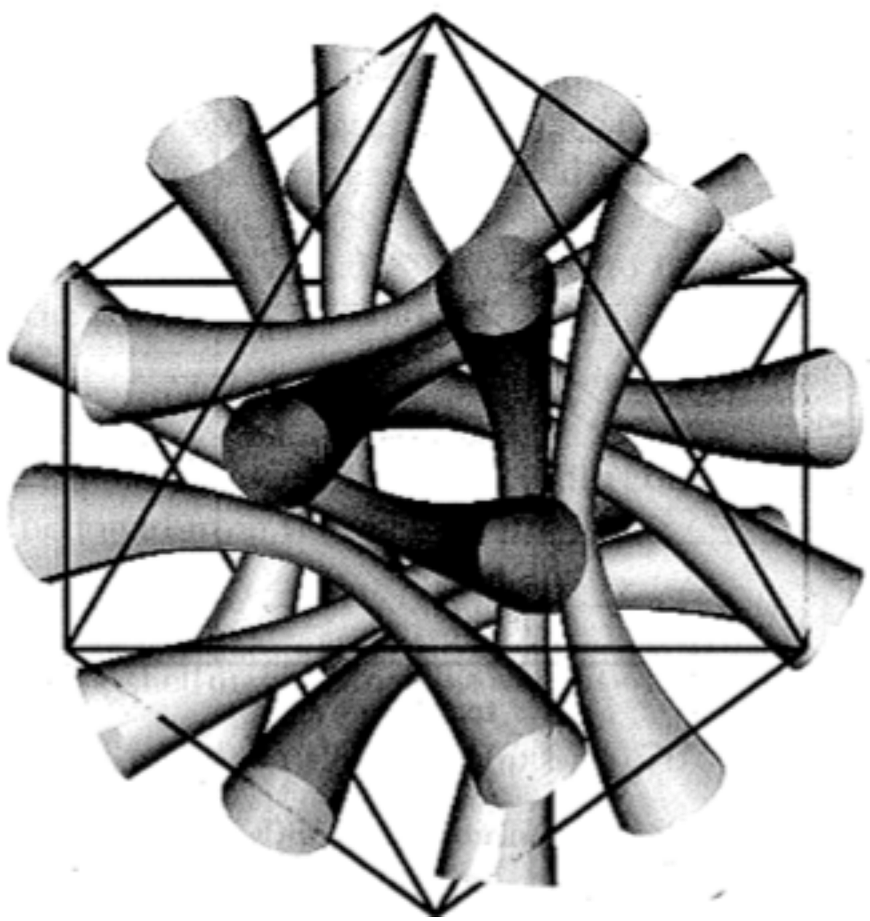
- idea: two vortex pairs propagate towards, and stretch, each other
- vorticity intensified, feedback to faster evolution

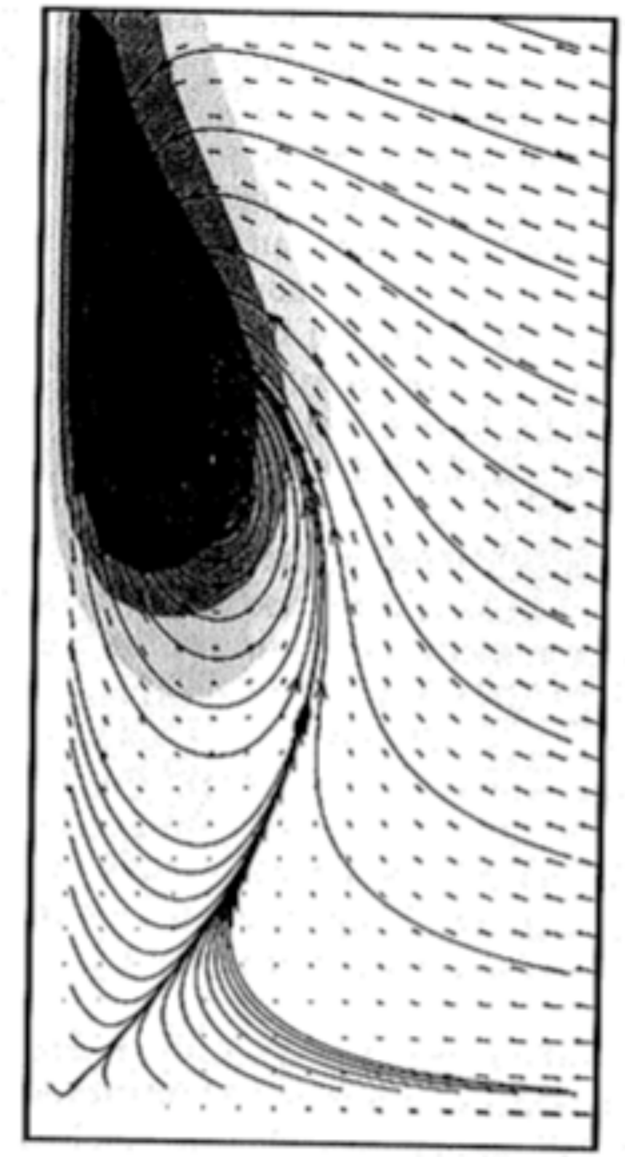


- singularity? not clear ; viscosity may not stop a singularity if it occurs

Colliding vortex pairs: Pelz

- 8 pairs colliding; highly symmetrical flow





/orticity contours (5,10,15,20 from light to dark) on the symmetry plane with velocity vectors and streamlines.

- using vortex filaments under Biot-Savart blow-up very clean

- but actual vortices tend to flatten, *depleting nonlinearity* in simulations

Evolution of anti-parallel vortices: Kerr/Bustamante

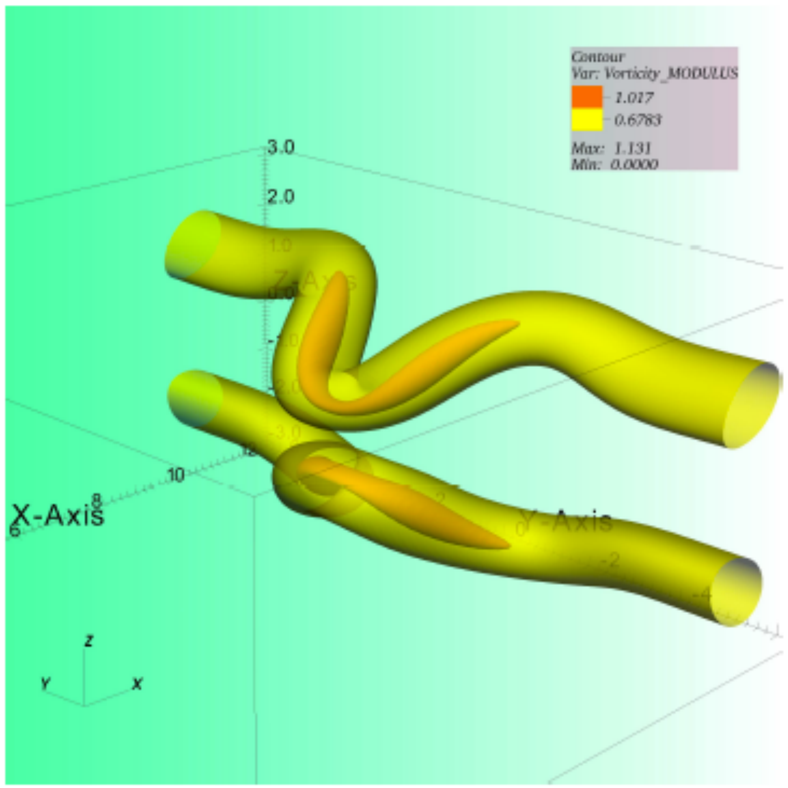


FIG. 5: Euler anti-parallel vortices in full periodic domain near $t = 2.51$. Bright (yellow online) tubes are isosurface contours of vorticity modulus corresponding to 60% of the instantaneous maximum of vorticity modulus. Dark (red online) elongated blobs are isosurfaces corresponding to 90% of the maximum of vorticity modulus.

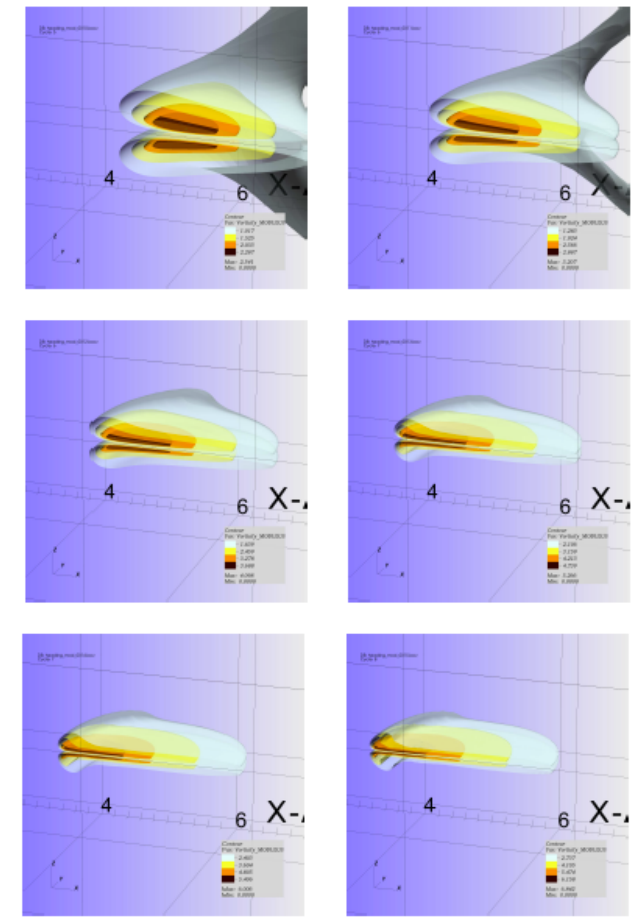
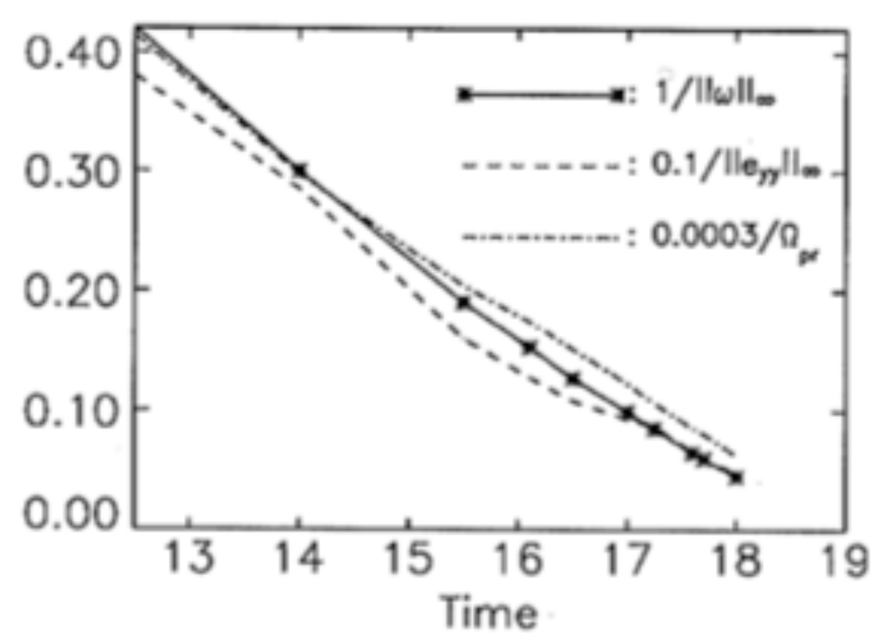


FIG. 6: From left to right, and from top to bottom: six successive, zoomed snapshots of the Euler anti-parallel vortices at times $t = 5.625, 6.25, 6.875, 7.5, 7.8125, 8.125$. The contours are sectioned through the $y = 0$ symmetry plane, to facilitate the view of the structures. The contours are isosurfaces of vorticity modulus corresponding, respectively from outer to inner, to the 40%, 60%, 80% and 90% of the value of the instantaneous maximum vorticity modulus.

- vorticity intensifies strongly
- and flattens to form tadpole structures
- singularity at t* = 18.7 ?



Vortex ring collisions in three dimensions

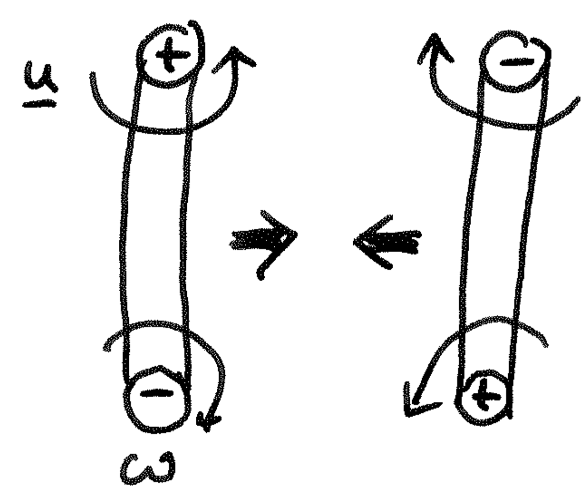
$$\boldsymbol{u} = u\hat{\boldsymbol{r}} + v\hat{\boldsymbol{z}}$$

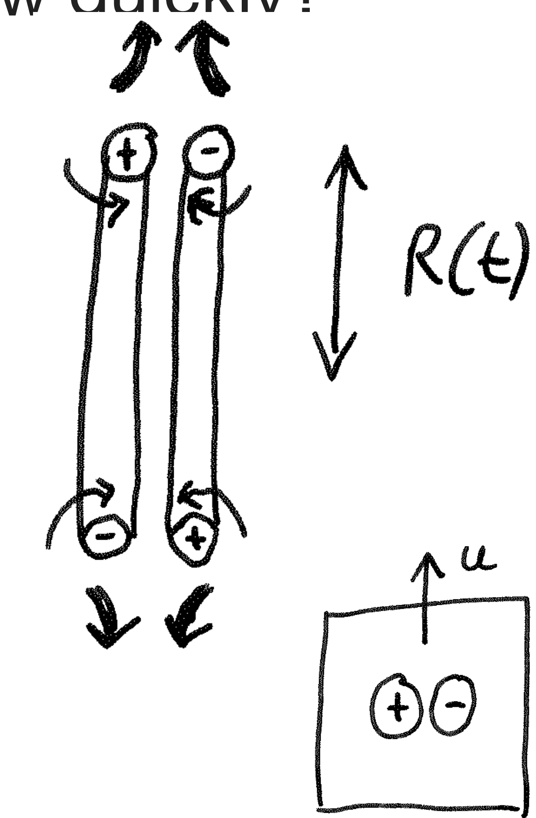
$$\boldsymbol{\omega} = \omega\hat{\boldsymbol{\theta}}$$

- https://www.youtube.com/watch?v=XJk8ijAUCiI
- https://www.youtube.com/watch?v=USzOciNHeh0&t=182s
- vortex rings move with the fluid (Helmholtz)
- then stretch (vortex line stretching) and accelerate outwards - how quickly?
- geometry:

$$(\partial_t + u\partial_r + v\partial_z)(\omega/r) = 0, \quad \omega = \partial_r v - \partial_z u,$$

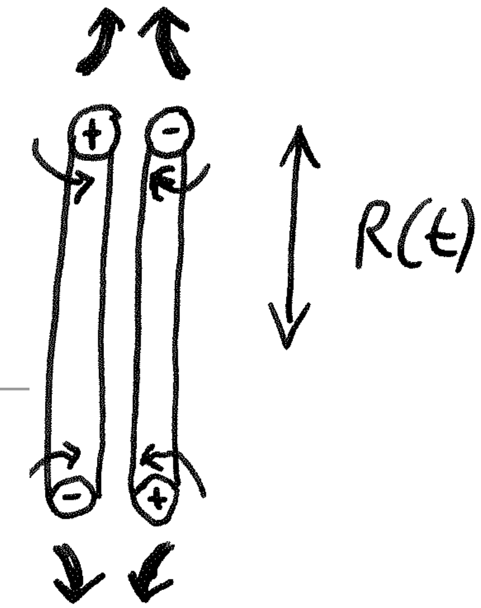
$$r^{-1}\partial_r(ru) + \partial_z v = 0$$

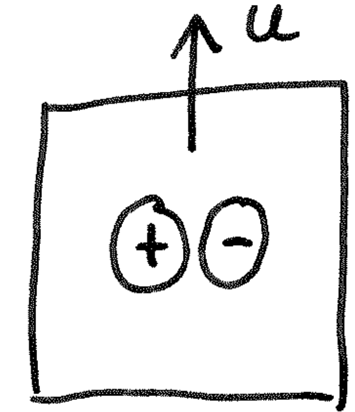




- approximate 2-d dipole travelling outwards

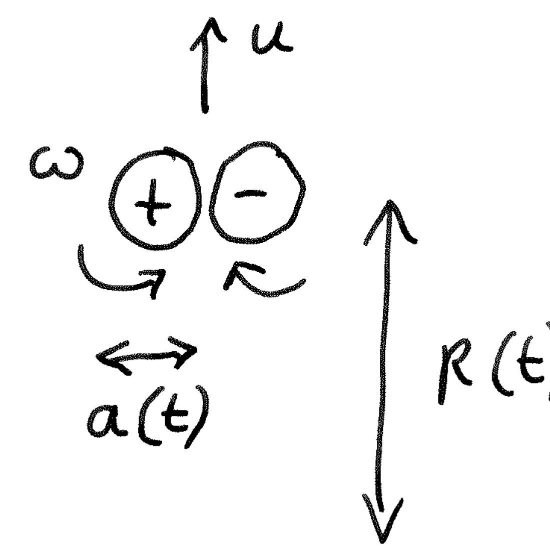
Theoretical ideas





- Childress, G, Valiant 2016

- major and minor axes $R(t), a(t)$



$$\text{vorticity} \quad \omega \sim R$$

$$\text{velocity} \quad \frac{dR}{dt} = u \sim a\omega \sim aR$$

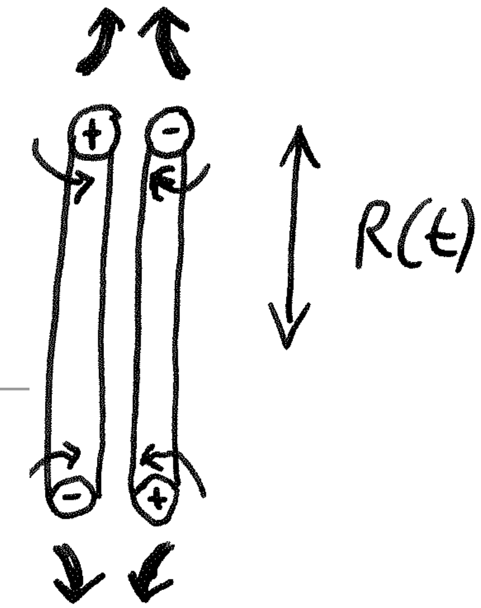
$$\text{volume} \quad V \sim a^2 R$$

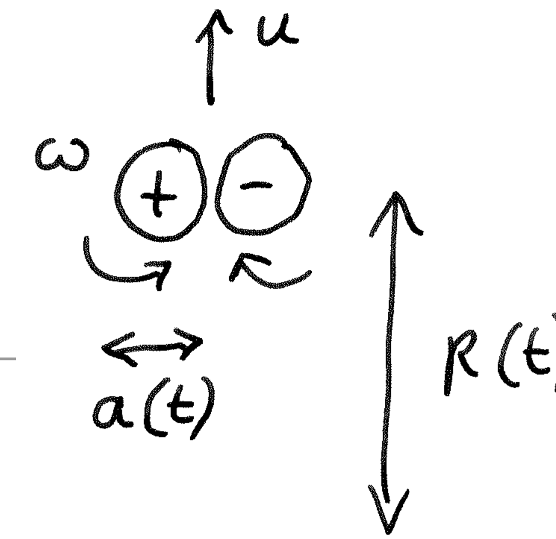
$$\text{energy} \quad E \sim Vu^2 \sim (a^2R)(aR)^2 = a^4R^3$$

- conserve volume: $a \sim R^{-1/2}$ $\quad \frac{dR}{dt} \sim R^{1/2}$ $\quad R^{1/2} \sim t, \quad R(t) \sim t^2, \quad a(t) \sim t^{-1}$

- problem… energy diverges: $E \sim t^2$

Theoretical ideas





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- major and minor axes $R(t), a(t)$

$$\text{vorticity} \quad \omega \sim R$$

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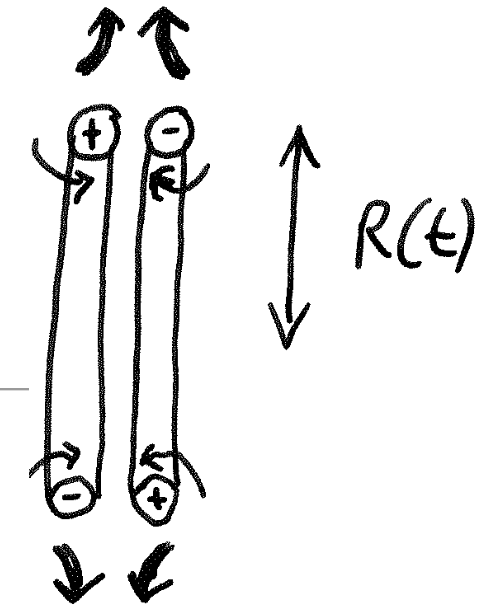
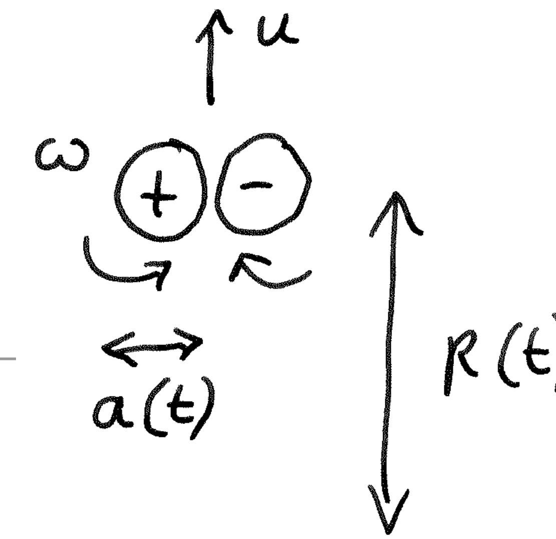
$$\text{energy} \quad E \sim Vu^2 \sim (a^2R)(aR)^2 = a^4R^3$$

- conserve energy: $a \sim R^{-3/4} \qquad \dfrac{dR}{dt} \sim R^{1/4} \qquad R^{3/4} \sim t, \quad R(t) \sim t^{4/3}, \quad a(t) \sim t^{-1}$

- necessarily, volume goes down, vorticity shed $\quad V \sim a^2 R = t^{-2/3}$

Vorticity shedding



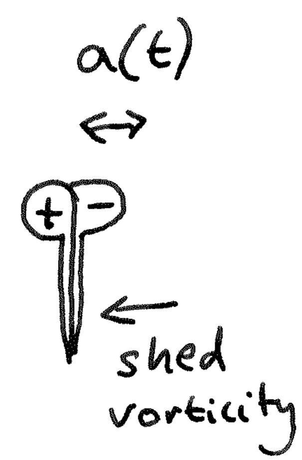


- original picture a bit naive
- conserve energy: energy $E \sim Vu^2 \sim (a^2R)(aR)^2 = a^4R^3$
- necessarily, volume goes down volume $V \sim a^2R$

$$R^{3/4} \sim t, \quad R(t) \sim t^{4/3}, \quad a(t) \sim t^{-1}$$

$$V \sim a^2R = t^{-2/3}$$

- must `lose' volume: shedding of vorticity in a tail behind the propagating vortex ring pair (visible on movies)
- `tadpole' or `snail' structure emerges



Simulations

- in axisymmetric flow
- only one ring shown

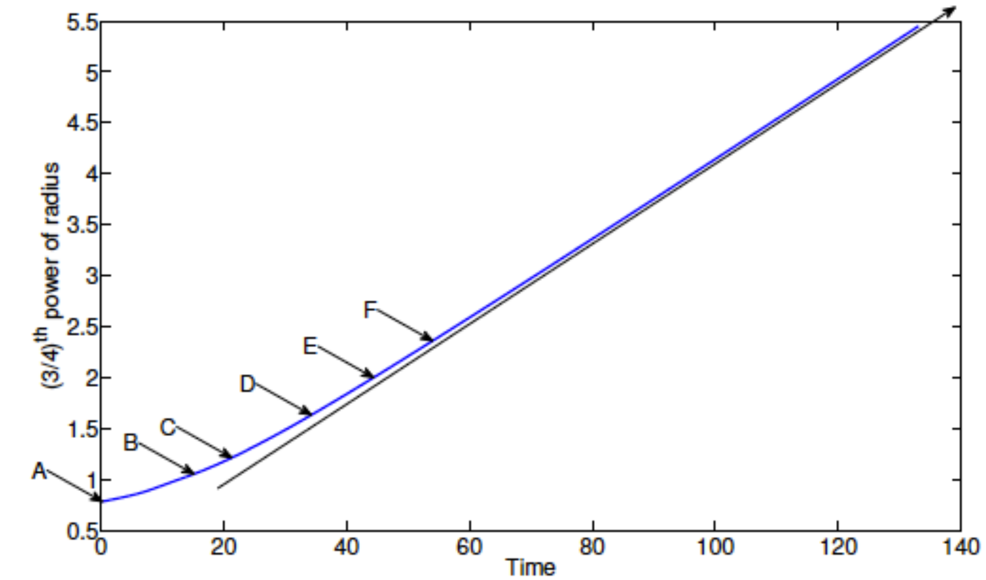


FIGURE 8. $R^{3/4}$ versus time, compared with a linear asymptote. The letters correspond to the frames of figure 7.

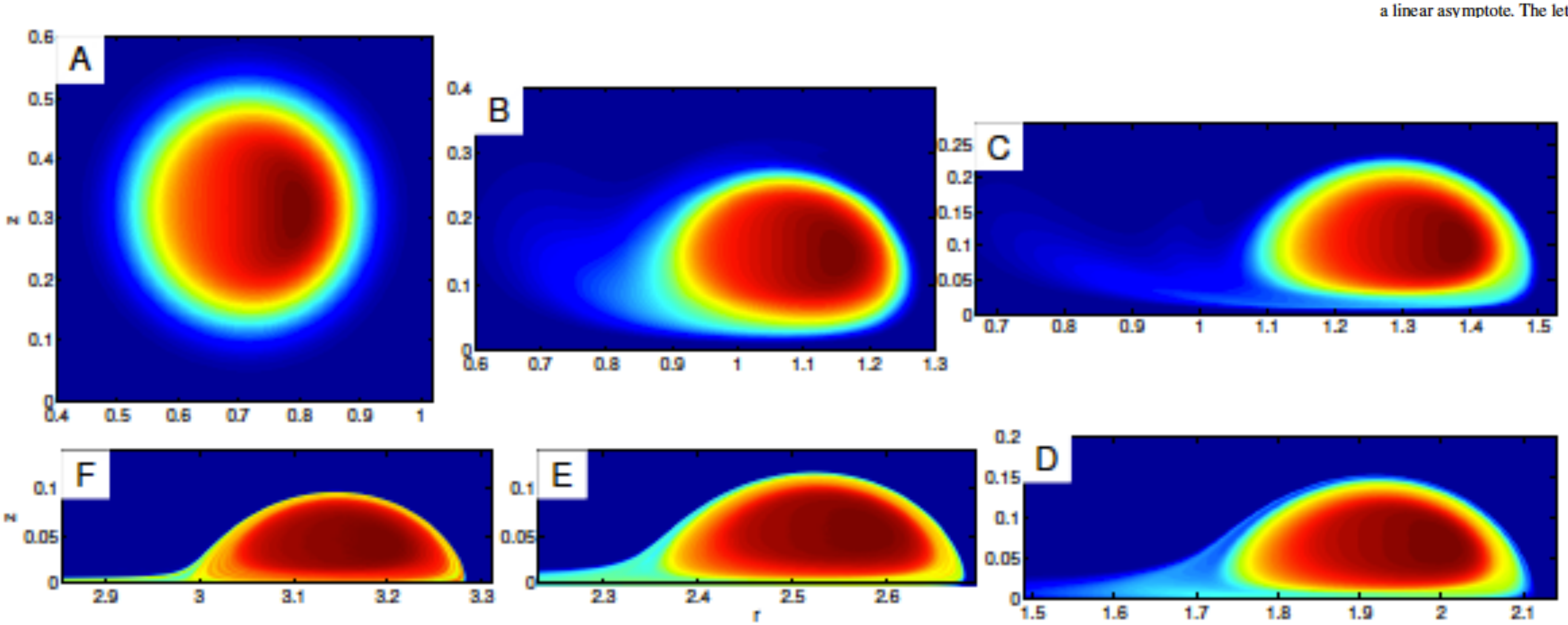
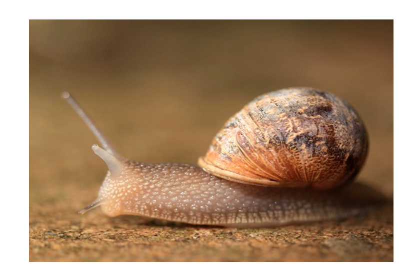
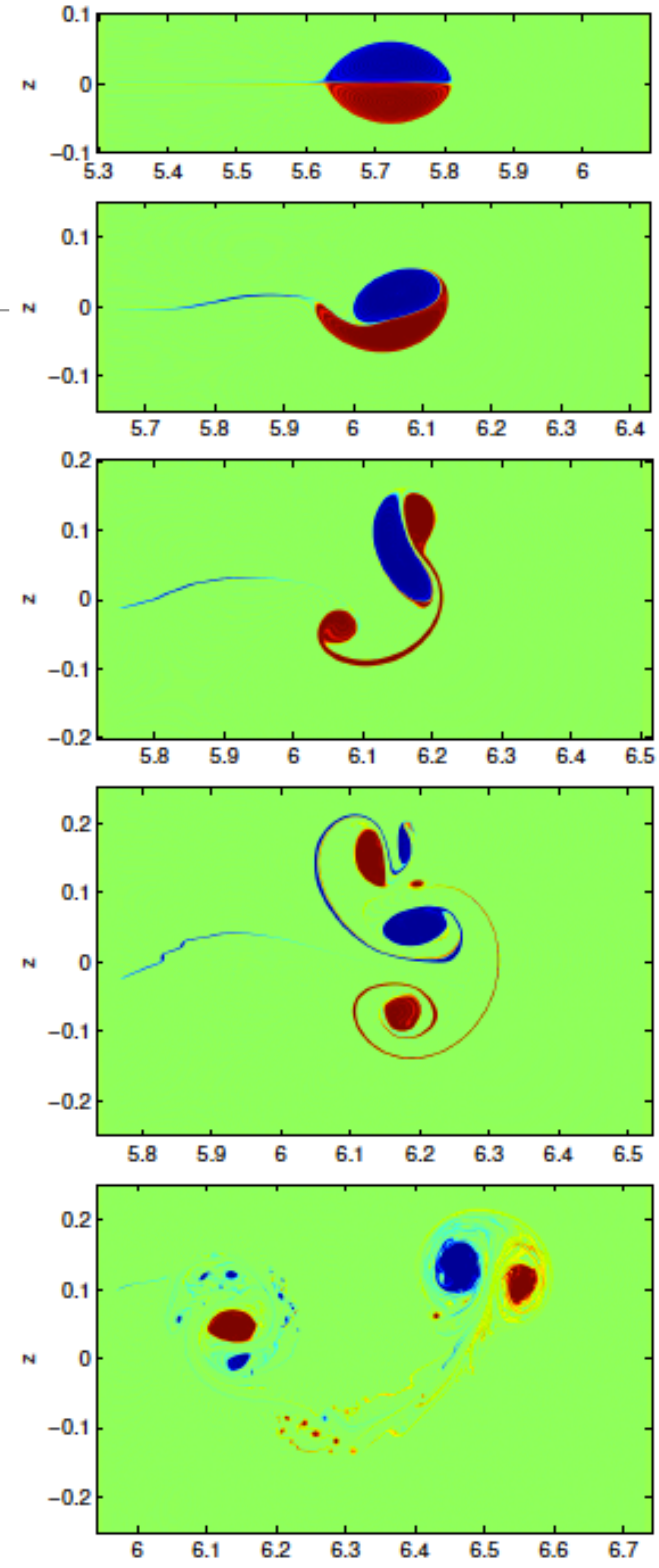


FIGURE 7. Development of the snail, shown at times t=0, 15, 21.2, 34.2, 44.3, 54, clockwise from the upper left corner. vorticity ω is plotted in the (r,z) plane at each time, scaled on the maximum value (red) in each plot, with zero blue.



Loss of symmetry

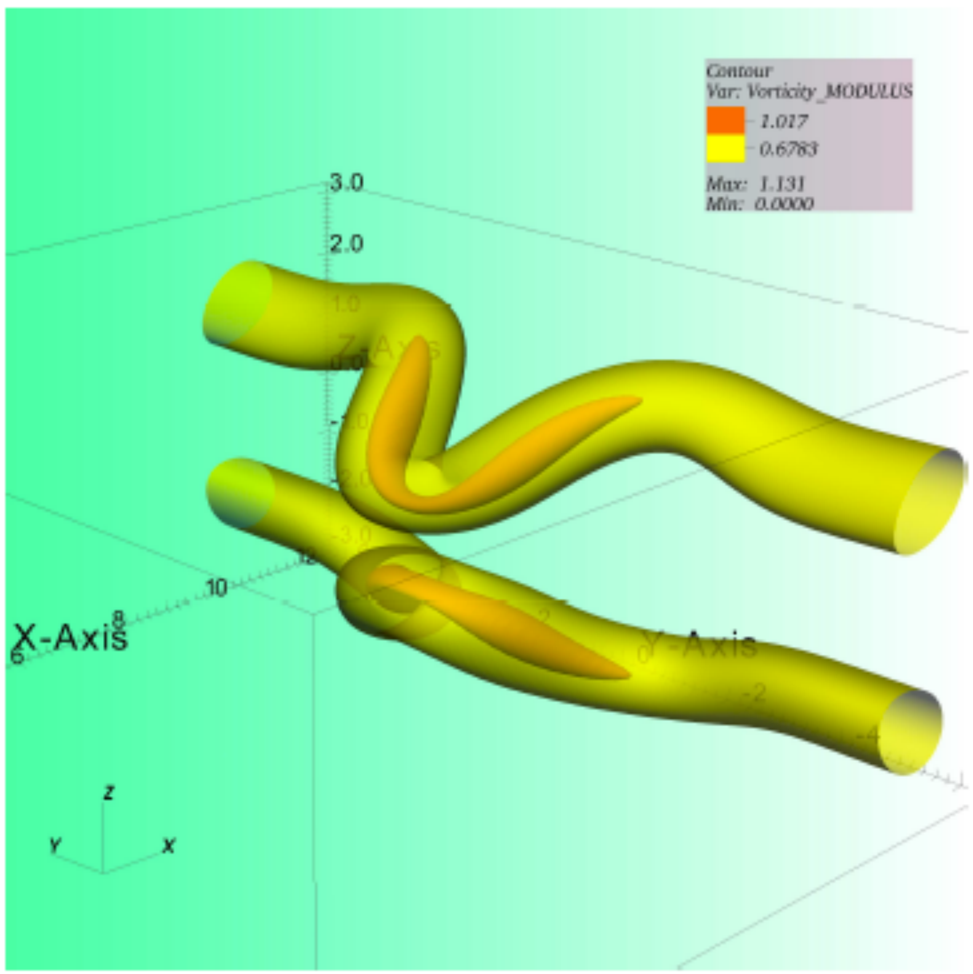
- up/down symmetry can be lost:



- also experiments reveal instabilities

More general geometry

- Bustamante & Kerr 2008



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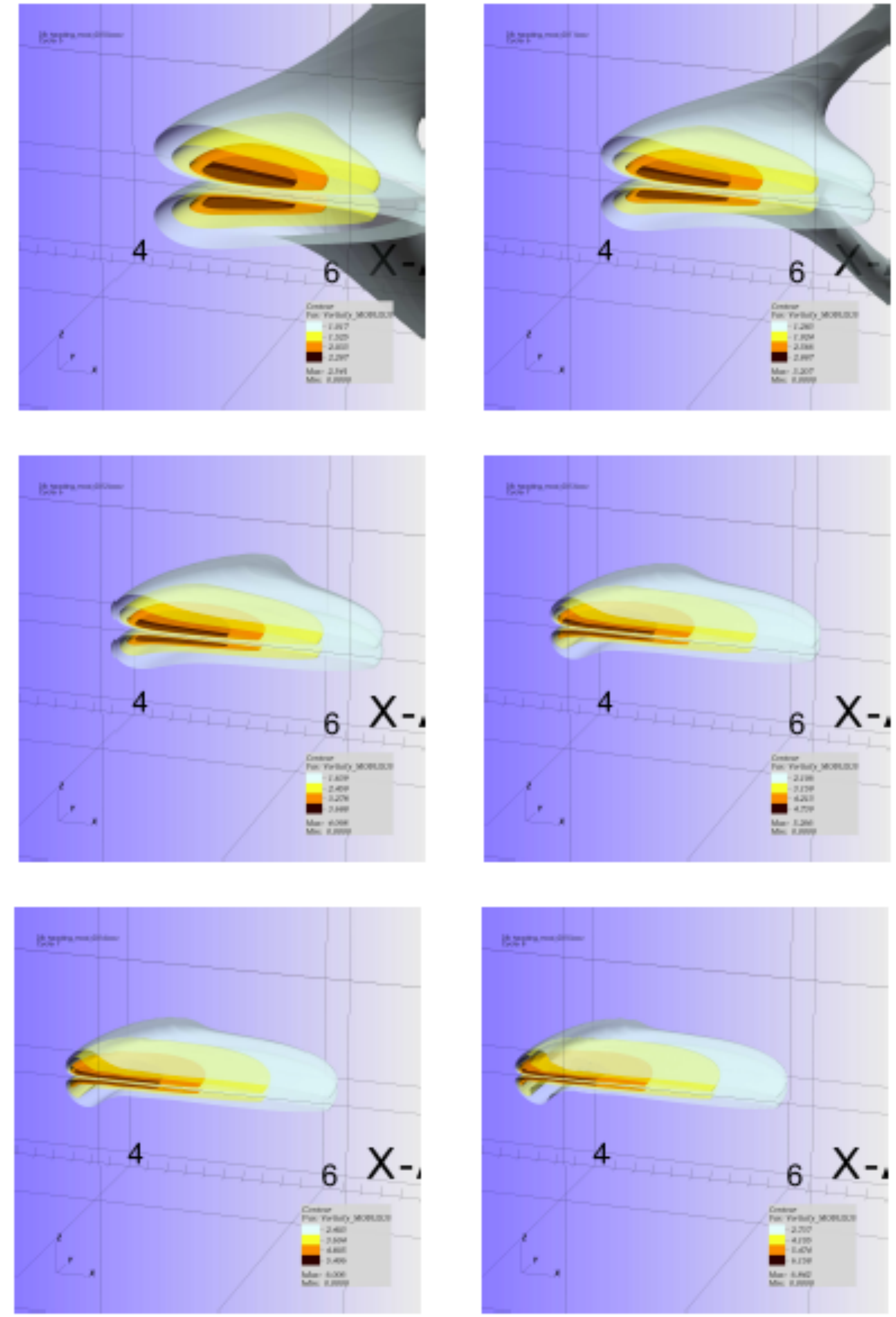

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Conclusions

- vorticity perhaps best way to understand nearly inviscid flows
- many challenges both for mathematics and analysing physical processes
- such as stretching and reconnection
- with links to outstanding theoretical issues such as the finite-time singularity question ….
- … and the nature of turbulence.

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