Spectral theory of soliton and breather gases in the focusing NLS equation

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(joint work with **Alexander Tovbis**, Central Florida)

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Waves, Coherent Structures, and Turbulence, UEA

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Motivation

V.E. Zakharov (SAPM, 2009): Turbulence in Integrable systems.

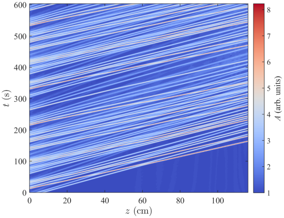
- **Mathematically: theory of integrable nonlinear PDEs with random initial or boundary conditions.**
- **1D conservative models. No vortices or cascades, sorry! No thermalisation either...**
- **Solitons and breathers are "particles" of integrable dispersive hydrodynamics.**
- **Hence the interest in soliton/breather gases—statistical ensembles of interacting solitons/breathers—a particular case of integrable turbulence.**

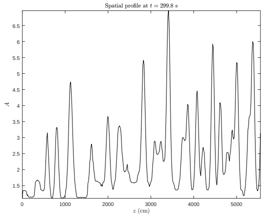
Example 1. Soliton gas in viscous fluid conduits

- interfacial dynamics of two immiscible buoyant viscous fluids;
- conduit equation: $A_t + (A^2)_z - (A^2(A^{-1}A_t)_z)_z = 0$.
- non-integrable, but soliton collisions are nearly elastic
  (Lowman, Hoefer and El, JFM 2014)

Soliton gas is created by a random input profile at nozzle

(Experiment at the Dispersive Hydrodynamics Laboratory at the University of Colorado, Boulder; M. Hoefer and M. Maiden)





Example 2: Shallow-water soliton gas

PRL 113, 108501 (2014) PHYSICAL REVIEW LETTERS week ending 5 SEPTEMBER 2014

Soliton Turbulence in Shallow Water Ocean Surface Waves

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We analyze shallow water wind waves in Currituck Sound, North Carolina and experimentally confirm, for the first time, the presence of soliton turbulence in ocean waves. Soliton turbulence is an exotic form of nonlinear wave motion where low frequency energy may also be viewed as a dense soliton gas, described theoretically by the soliton limit of the Korteweg–deVries equation, a completely integrable soliton system: Hence the phrase "soliton turbulence" is synonymous with "integrable soliton turbulence." For periodic-quasiperiodic boundary conditions the ergodic solutions of Korteweg–deVries are exactly solvable by finite gap theory (FGT), the basis of our data analysis. We find that large amplitude measured wave trains near the energetic peak of a storm have low frequency power spectra that behave as $\sim\omega^{-1}$. We use the linear Fourier transform to estimate this power law from the power spectrum and to filter densely packed soliton wave trains from the data. We apply FGT to determine the soliton spectrum and find that the low frequency $\sim\omega^{-1}$ region is soliton dominated. The solitons have random FGT phases, a soliton random phase approximation, which supports our interpretation of the data as soliton turbulence. From the probability density of the solitons we are able to demonstrate that the solitons are dense in time and highly non-Gaussian.

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PHYSICAL REVIEW LETTERS 122, 214502 (2019)

Editors' Suggestion Featured in Physics

Experimental Evidence of a Hydrodynamic Soliton Gas

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(Received 29 November 2018; published 29 May 2019)

We report on an experimental realization of a bidirectional soliton gas in a 34-m-long wave flume in a shallow water regime. We take advantage of the fission of a sinusoidal wave to continuously inject solitons that propagate along the tank, back and forth. Despite the unavoidable damping, solitons retain their profile adiabatically, while decaying. The outcome is the formation of a stationary state characterized by a dense soliton gas whose statistical properties are well described by a pure integrable dynamics. The basic ingredient in the gas, i.e., the two-soliton interaction, is studied in detail and compared favorably with the analytical solutions of the Kaup-Boussinesq integrable equation. High resolution space-time measurements of the surface elevation in the wave flume provide a unique tool for studying experimentally the whole spectrum of excitations.

Example 3: Breather gas in the ocean (NLS)

Ocean Dynamics
https://doi.org/10.1007/s10236-018-1232-y

CrossMark

Highly nonlinear wind waves in Currituck Sound: dense breather turbulence in random ocean waves

Alfred R. Osborne[1] · Donald T. Resio[2] · Andrea Costa[3,4] · Sonia Ponce de León[5] · Elisabetta Chirivì[6]

Ocean Dynamics

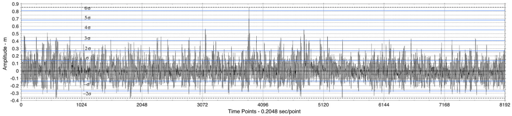


Fig. 17 Time series of 8192 points from Currituck Sound at 21:00 on 4 February 2002. The length of the time series is 1677.72 s = 27.962 min and the discretization interval is 0.2048 s. The standard deviation is $\sigma = 13.7$ cm, the significant wave height is $H_s = 4\sigma = 54.7$ cm, the peak period is $T_p = 2.51$ s (spectral average over 9 probes) and the zero crossing period is $T_z = 2.38$ s, giving 705 zero crossing waves. The blue horizontal lines correspond to the number of standard deviations above and below the zero mean. The largest measured wave amplitude is 86 cm (over six standard deviations tall) and the largest wave height (the same wave) is 114 cm, which corresponds to $2.08H_s$

Outline of the talk

- **Kinetic equation for soliton gas: an elementary construction**
- **Finite-gap potentials and nonlinear dispersion relations**
- **Thermodynamic limit and the equation of state of breather/soliton gas**
- **Ideal soliton/breather gas and soliton condensate**
- **Kinetic equation for breather/soliton gas and particular solutions**

Soliton gas: an elementary construction

Rarefied gas of KdV solitons *(Zakharov, JETP 1971)*

Starting point: N**-soliton solution** $u_N(x,t)$ **of the KdV equation**

$$u_t + 6uu_x + u_{xxx} = 0\,.$$

If solitons are sufficiently separated, then u_N can be locally approximated by a superposition of N single KdV solitons. Consider a random process:

$$u_\infty = \sum_{i=1}^{\infty} 2\eta_i^2 \mathrm{sech}^2[\eta_i(x - 4\eta_i^2 t - x_i)],$$

characterised by two distributions:

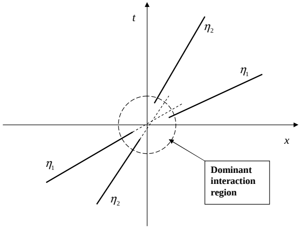
1. **Spectral distribution function** (density of states) $f(\eta)$: the number of solitons with $\eta_i \in [\eta_0, \eta_0 + d\eta]$ per unit interval of x is $f(\eta_0)d\eta$.
2. **Poisson distribution** for $x_i \in \mathbb{R}$ with small density $\int f(\eta)d\eta \ll 1$.

Properties of soliton collisions

- Isospectrality $(d\eta_i/dt = 0) \implies$ elastic collisions;
- Phase shifts.

Phase (position) shifts

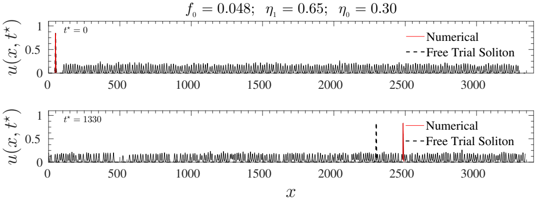
- Solitons interact pairwise (multi-particle effects are absent);
- Each collision gives rise to phase shifts of the interacting solitons.



For a two-soliton collision with $\eta_1 > \eta_2$ the phase shifts as $t \to +\infty$ are

$$\delta_1 = \frac{1}{\eta_1} \ln\left(\frac{\eta_1 + \eta_2}{\eta_1 - \eta_2}\right), \quad \delta_2 = -\frac{1}{\eta_2} \ln\left(\frac{\eta_1 + \eta_2}{\eta_1 - \eta_2}\right).$$

Kinetic equation for a rarified soliton gas *(Zakharov, JETP 1971)*



- Let $\eta \in [0, 1]$ and $\rho = \int_0^1 f(\eta) d\eta \ll 1$. Then the speed of a "trial" η-soliton in a soliton gas with the distribution function $f(\eta)$:

$$s(\eta) = 4\eta^2 + \frac{1}{\eta} \int_0^1 \ln \left| \frac{\eta + \mu}{\eta - \mu} \right| f(\mu)[4\eta^2 - 4\mu^2] d\mu + o(\rho) \qquad (1)$$

- Consider now a spatially non-homogeneous soliton gas. Assume

$$f(\eta) \equiv f(\eta; x, t), \qquad s(\eta) \equiv s(\eta; x, t); \qquad \Delta x, \Delta t \gg 1.$$

Then isospectrality of the KdV dynamics implies:

$$f_t + (sf)_x = 0, \qquad (2)$$

- Equations (2), (1) form the **kinetic equation** for a rarefied soliton gas.

Kinetic equation for a dense soliton gas: KdV

Kinetic equation for a dense KdV soliton gas as the thermodynamic limit of the **KdV-Whitham modulation equations** *(El, Phys Lett A, 2003)*

$$f_t + (fs)_x = 0, \tag{3}$$

$$s(\eta) = 4\eta^2 + \frac{1}{\eta}\int_0^1 \ln\left|\frac{\eta+\mu}{\eta-\mu}\right| f(\mu)[s(\eta)-s(\mu)]\,d\mu. \tag{4}$$

- A nonlinear integro-differential equation
- Suggests a general recipe for the construction of soliton kinetic equations for other integrable PDEs via the phase-shift kernel *(El and Kamchatnov, PRL 2005)*. (Watch out for the talk of T. Congy!)
- Recently derived from a completely different perspective for quantum many-body integrable systems (B. Doyon et. al. PRL (2018) . . .)

Spectral theory of breather/soliton gas

in the focusing NLS equation

Spectral theory of soliton/breather gas: High Level Description

- Kinematic theory of linear dispersive waves (Whitham)
  $\psi \sim a(x,t)e^{i\theta(x,t)}$, $k = \theta_x$, $\omega = \theta_t$

$$k_t = \omega_x; \quad \omega = \omega_0(k)$$

- An analogue for n-phase nonlinear waves $\psi = \Psi(\theta_1, \dots, \theta_n)$:

$$\mathbf{k}_t = \boldsymbol{\omega}_x; \quad \mathbf{k} = (k_1, \dots, k_n), \quad \boldsymbol{\omega} = (\omega_1, \dots, \omega_n).$$

  Nonlinear dispersion relations:

$$\mathbf{k} = \mathbf{K}(\Sigma_n), \quad \boldsymbol{\omega} = \boldsymbol{\Omega}(\Sigma_n),$$

  where Σ_n is the "nonlinear Fourier" (IST) band spectrum.

- For a special "thermodynamic" scaling of Σ_n, the limit $n \to \infty$ yields the kinetic equation for the density of states $u(\eta, x, t)$

$$u_t + (us)_x = 0, \quad s(\eta, x, t) = \mathcal{F}[u(\eta, x, t)],$$

  where $\eta \in \mathbb{C}$, and the functional $\mathcal{F}$ specifies the "equation of state" for a soliton (breather) gas.

Focusing NLS equation: spectral problem

$$i\psi_t + \psi_{xx} + 2|\psi|^2\psi = 0\,.$$

The IST method (*Zakharov and Shabat 1972*) links the NLS time evolution with the time evolution of the scattering data of the linear ZS equation

$$\begin{pmatrix} \partial_x + i\lambda & -\psi(x,t) \\ -\psi^*(x,t) & \partial_x - i\lambda \end{pmatrix} Y = \mathcal{L}^{(x)} Y = 0,$$

where $\psi(x,t)$ is the NLS solution, $\lambda \in \mathbb{C}$ is the spectral parameter, $Y = Y(x,t,\lambda) \in \mathbb{C}^2$.

The spectrum of ψ: $\Sigma(\psi) = \{\lambda \in \mathbb{C} | \mathcal{L}^{(x)} Y = 0, |Y| < \infty \ \forall x\}$

- Decaying potentials: the spectrum $\Sigma(\psi)$ generally has two components: discrete (solitons) and continuous (dispersive radiation).
- Finite-band (finite-gap) potentials ψ_n: $\Sigma_n(\psi) = \cup_{i=0}^{n} \gamma_i$.

— Multi-phase periodic or quasiperiodic solutions.

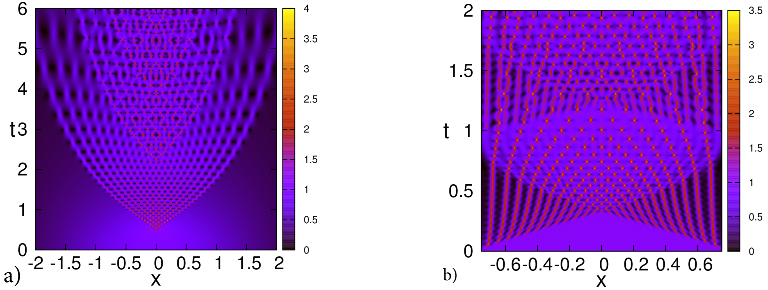
$$\psi_n = \Psi(\theta_1, \dots, \theta_n), \quad \Psi(\dots, \theta_j + 2\pi, \dots) = \Psi(\dots, \theta_j + 2\pi, \dots).$$

$$\theta_j = k_j x + \omega_j + \theta_j^{(0)}$$

—Solitons and breathers are some limiting cases of finite-gap potentials

Emergence of finite-gap solutions in semi-classical evolution

$$i\varepsilon\psi_t + \frac{\varepsilon^2}{2}\psi_{xx} + 2|\psi|^2\psi = 0, \qquad \varepsilon \ll 1.$$

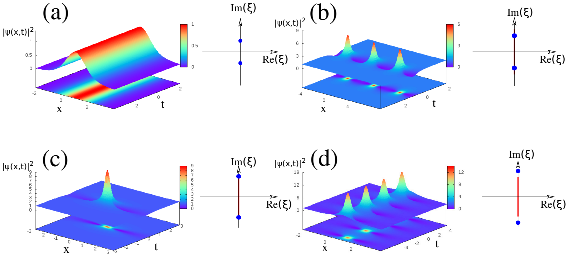


El, Khamis and Tovbis, Nonlinearity (2016)

- The solution is locally approximated by finite-gap potentials ψ_n.
- The genus (the number of nonlinear oscillatory modes n) increases with time.
- Soliton gas at $t \gg 1$.
  Optics experiment: *G. Marcucci et al, Nature Comm. (2019)*

Spectral portraits of NLS solitons and "standard" breathers

IST spectral parameter $\xi \in \mathbb{C}$

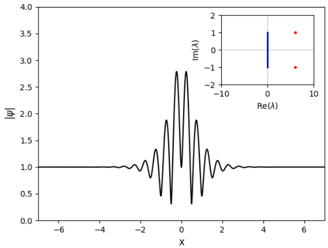


- (a) fundamental soliton

$$\psi_S(x,t) = 2ib\,\mathrm{sech}[2b(x+4at-x_0)]e^{-2i(ax+2(a^2-b^2)t)+i\phi_0}.$$

(b) Akhmediev breather; (c) Peregrine soliton;
(d) Kuznetsov-Ma breather.

Tajiri-Watanabe (TW) breather



Two velocities associated with the TW breather

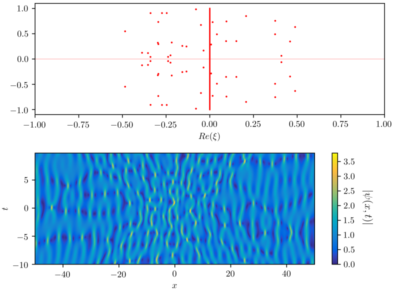
$$c_g = -2\frac{\Im[\lambda R_0(\lambda)]}{\Im R_0(\lambda)} \equiv s_{TW}(\lambda), \quad c_p = -\frac{2\Re[\lambda R_0(\lambda)]}{\Re R_0(\lambda)},$$

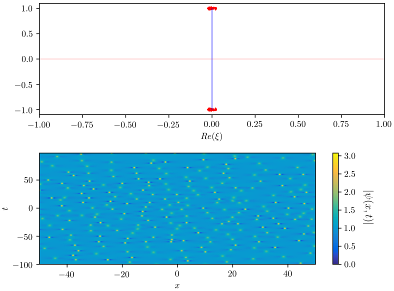
$$\text{where } R_0(\lambda) = \sqrt{\lambda^2 + q^2}.$$

- Akhmediev, Kuznetsov-Ma and Peregrine breathers are particular cases of the TW breather with the double points $\lambda, \overline{\lambda}$ of the spectrum located on the imaginary axis.

Fundamental solitons (TW breathers) are the "particles" in a soliton (breather) gas.

Breather gas examples: “Akhmediev-like” and “Peregrine-like” gases





Finte-Gap NLS solutions

and Nonlinear Dispersion Relations

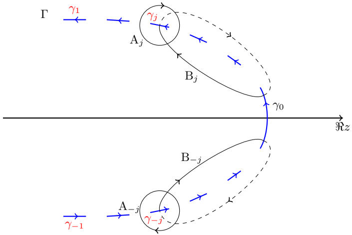
Finite-gap NLS solutions: basic configuration

- Focusing NLS $i\psi_t + \psi_{xx} + 2|\psi|^2\psi = 0$,
- Finite-gap solutions ψ_n live on a hyperelliptic **genus** n Riemann surface of

$$R(z) = \prod_{j=0}^{n}(z-\alpha_j)^{1/2}(z-\bar{\alpha}_j)^{1/2}, \quad \alpha_j = a_j + ib_j, \quad b_j > 0,$$

$z \in \mathbb{C}$: the spectral parameter in the Zakharov-Shabat scattering problem.

- Assume even genus $n = 2N$, $N \in \mathbb{N}$.
- Let all bands lie on a 1D Schwarz-symmetrical curve Γ.



- Exceptional (Stokes) band γ_0 and regular ("solitonic") bands $\gamma_{\pm j}$, $j = 1, \ldots, N$. Can have several Stokes bands.
- Transition to an odd genus n via closing the Stokes band γ_0

Wavenumbers and frequencies

- Introduce a special wavenumber-frequency set:

  $\mathbf{k} = (k_1, \ldots, k_N, \tilde{k}_1, \ldots, \tilde{k}_N)$, $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_N, \tilde{\omega}_1, \ldots, \tilde{\omega}_N)$

- Contrasting behaviours for "solitonic" (k_j, ω_j) and "carrier" $(\tilde{k}_j, \tilde{\omega}_j)$ components:

  Soliton/breather limit: **collapse a band into a double point**:

$$\alpha_{2j} \to \alpha_{2j+1} \ (|\gamma_j| \to 0)$$

$$\Longrightarrow$$

$$k_j, \omega_j \to 0, \quad \tilde{k}_j, \tilde{\omega}_j = O(1)$$

A soliton (breather) on the finite-gap potential background.

Nonlinear dispersion relations for finite-gap NLS solutions

For solitonic components of $\mathbf{k}$, $\boldsymbol{\omega}$ we obtain **nonlinear dispersion relations** $k_j = k_j(\boldsymbol{\alpha})$, $\omega_j = \omega_j(\boldsymbol{\alpha})$ (cf. *Flaschka, Forest, McLaughlin CPAM, (1982) for KdV*):

$$\begin{aligned}
\sum_{|m|=1}^{N} k_m \Im \oint_{\mathrm{B}_m} \frac{P_j(\zeta)d\zeta}{R(\zeta)} &= \pi \Re \varkappa_{j,1}, \\
\sum_{|m|=1}^{N} \omega_m \Im \oint_{\mathrm{B}_m} \frac{P_j(\zeta)d\zeta}{R(\zeta)} &= 2\pi \Re (\varkappa_{j,1} \sum_{k=1}^{2N+1} \Re \alpha_k + \varkappa_{j,2}), \\
|j| = 1, \dots, N,
\end{aligned}$$

where

$$P_j(z) = \varkappa_{j,1} z^{2N-1} + \varkappa_{j,2} z^{2M-2} + \cdots + \varkappa_{j,2N}$$

$$R(z) = \prod_{|j|=0}^{N} (z - \alpha_{2j})^{1/2} (z - \alpha_{2j+1})^{1/2}$$

and $\varkappa_{i,j}$ are the coefficients of the normalised holomorphic differentials:

$$w_j = [P_j(z)/R(z)]dz, \quad \oint_{\mathrm{A}_i} w_j = \delta_{ij}, \quad i,j = 1, \dots, N.$$

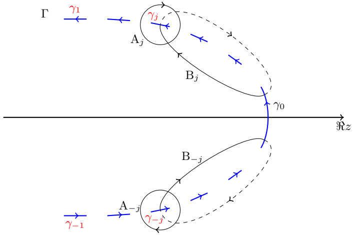
Similar nonlinear dispersion relations exist for the carrier components $\tilde{k}_i$ and $\tilde{\omega}_i$.

Thermodynamic limit of finite-gap solutions

(Soliton/Breather gas)

Thermodynamic limit

We are interested in a special, large N limit so that $\forall k_j \to 0$ but $\lim_{N\to\infty} \sum_{j=1}^{N} k_j = \mathcal{O}(1)$ — the thermodynamic limit



- Introduce

$$\eta_j = \frac{1}{2}(\alpha_{2j+1} + \alpha_{2j}), \qquad \delta_j = \frac{1}{2}(\alpha_{2j} - \alpha_{2j+1}), \qquad |j| = 1, \ldots, N,$$

  η_j are the centres of the bands γ_j and $2|\delta_j|$ the bandwidths.

- For the exceptional (Stokes) band γ_0 we have $\delta_0 = \frac{1}{2}(\alpha_1 - \alpha_{-1})$.

Three spectral scalings

Let $N \gg 1$ and assume:

- the band centres η_j are distributed along the curve Γ with some limiting density $\varphi(\eta) > 0$, $\eta \in \Gamma$.
- $|\eta_j - \eta_{j+1}| = O(1/N)$.

Options for the scaling of the spectral bandwidth $|\delta_j|$:

(i) <u>*Exponential scaling*</u> (general):

$$|\delta_j| \sim e^{-N\tau(\eta_j)},$$

where $\tau(\mu)$ is a smooth positive function on Γ.

(iii) <u>*Super-exponential scaling*</u> ("ideal gas"): for any $a > 0$

$$|\delta_j| \ll e^{-aN}$$

(ii) <u>*Sub-exponential scaling*</u> ("condensate"): for any $a > 0$

$$e^{-aN} \ll |\delta_j| \ll \frac{1}{N}$$

For all three scalings: $|\mathbf{gap}_j| = \mathcal{O}(1/N)$ **so** $|\mathbf{band}_j|/|\mathbf{gap}_j| \to 0$ **as** $N \to \infty$:
soliton/breather gas limits

Nonlinear dispersion relations for soliton gas

- Assume the **exponential spectral scaling** $|\delta_j| \sim e^{-N\tau(\eta_j)}$ so that for $N \gg 1$ the spectrum is characterised by two positive functions: $\varphi(\eta)$ and $\tau(\eta)$
- Introduce the scaling for solitonic wavenumbers and frequencies:

$$k_j = \frac{\kappa(\eta_j)}{N}, \qquad \omega_j = \frac{\nu(\eta_j)}{N}, \qquad N \gg 1,$$

where $\kappa(\eta), \nu(\eta) = \mathcal{O}(1)$ are continuous functions on Γ.

- Apply the limit $N \to \infty$ to the finite-gap nonlinear dispersion relations. For soliton gas we obtain (equations for breather gas have similar structure):

$$\int_{\Gamma^+} \ln \left| \frac{\mu - \bar{\eta}}{\mu - \eta} \right| u(\mu)|d\mu| + \sigma(\eta)u(\eta) = \pi \Im \eta,$$

$$\int_{\Gamma^+} \ln \left| \frac{\mu - \bar{\eta}}{\mu - \eta} \right| v(\mu)|d\mu| + \sigma(\eta)v(\eta) = 4\pi \Im \eta \Re \eta,$$

where

- $u(\eta) = \kappa(\eta)\varphi(\eta) > 0$ is the **density of states**,
- $v(\eta) = \nu(\eta)\varphi(\eta)$—its temporal counterpart,
- $\sigma(\eta) = \frac{2\tau(\eta)}{\varphi(\eta)} \geq 0$ is the **"spectral signature"** function.

Ideal gas and soliton/breather condensate

Consider the balance of terms in nonlinear dispersion relations for soliton gas

$$\int_{\Gamma^+} \ln\left|\frac{\mu-\bar{\eta}}{\mu-\eta}\right| u(\mu)|d\mu| + \sigma(\eta)u(\eta) = \pi\Im\eta,$$

$$\int_{\Gamma^+} \ln\left|\frac{\mu-\bar{\eta}}{\mu-\eta}\right| v(\mu)|d\mu| + \sigma(\eta)v(\eta) = 4\pi\Im\eta\Re\eta,$$

- $u \to 0$, $\sigma \to \infty$, $u\sigma = \mathcal{O}(1)$: **ideal gas** of non-interacting solitons (super-exponential spectral scaling); In this limit $s(\eta) = -v/u = -4\Re\eta$.
- $\sigma(\eta) \to 0$, $u(\eta) = \mathcal{O}(1)$: **"soliton condensate"** (sub-exponential scaling, interactions dominate). Fully defined by the spectral locus curve Γ.

Example: bound state soliton condensate

Bound states are N-soliton solutions, in which all solitons travel with the same speed V; w.l.o.g. $V = 0$.

- $\Gamma = [-iq, iq]$
- The nonlinear dispersion relations for a bound state soliton gas:

$$v(\eta) = 0, \tag{1}$$

$$\int_{-iq}^{iq} \ln \left| \frac{\mu - \bar{\eta}}{\mu - \eta} \right| u(\mu)(-id\mu) + \sigma(\eta)u(\eta) = \pi \Im \eta \tag{2}$$

- For the soliton condensate we have $\sigma = 0$ and Eq. (2) can be readily solved (finite Hilbert transform):

$$u_c(\eta) = \frac{-i\eta}{\pi\sqrt{\eta^2 + q^2}}, \quad \eta \in (-iq, iq). \tag{3}$$

Eq.(3) coincides with the normalised "Weyl" semi-classical distribution of discrete spectrum in a rectangular barrier (box) potential of the hight q.

Watch out for tomorrow's Pierre Suret talk on the bound state soliton condensate and MI.

Kinetic equation for soliton/breather gas

Equation of state for breather (soliton) gas

Eliminating $\sigma(\eta)$ from the nonlinear dispersion relations we obtain the **equation of state** for breather (soliton) gas

$$s(\eta) = s_0(\eta) + \int_{\Gamma^+} \Delta(\eta,\mu)[s(\mu) - s(\eta)]u(\mu)|d\mu|,$$

where $s(\eta) = -v(\eta)/u(\eta)$ is the "tracer" soliton (breather) velocity in a gas.

- For soliton gas:

$$s_0(\eta) = 4\Re\eta; \quad \Delta(\eta,\mu) = \frac{1}{\pi\Im\eta} \ln\left|\frac{\mu-\bar{\eta}}{\mu-\eta}\right|$$

- For breather gas:

$$s_0(\eta) = \Re\eta + \Im\eta\frac{\Re R_0(\eta)}{\Im R_0(\eta)} = s_{TW},$$

$$\Delta(\eta,\mu) = \frac{1}{\pi\Im R_0\eta)}\left[\ln\left|\frac{\mu-\bar{\eta}}{\mu-\eta}\right| + \ln\left|\frac{R_0(\eta)R_0(\mu)+\eta\mu-\delta_0^2}{R_0(\bar{\eta})R_0(\mu)+\bar{\eta}\mu-\delta_0^2}\right|\right]$$

Here $\Delta(\eta,\mu)$ is the **position shift for the 2-soliton (2-breather) interaction.**

The criticality (condensation) condition revisited

The equation for the density of states of a breather (soliton) condensate can be written as

$$\int_{\Gamma^+} \Delta(\eta, \mu) u(\mu) |d\mu| = 1, \qquad (*)$$

where $\Delta(\eta, \mu)$ is the position shift in the breather-breather (soliton-soliton) interactions, $\eta, \mu \in \Gamma^+$

- Integral (Fredholm 1st kind) equation for the critical density of states $u(\eta)$
- In the case of soliton gas can be solved for certain geometries of Γ^+

Kinetic equation

Consider a weakly non-homogeneous soliton/breather gas with $u = u(\eta, x, t)$, $s = s(\eta, x, t)$. Then it can be shown that the density of states satisfies

$$u_t + (us)_x = 0$$

Adding the equation of state

$$s(\eta) = s_0(\eta) + \int_{\Gamma^+} \Delta(\eta, \mu)[s(\mu) - s(\eta)]u(\mu)|d\mu|,$$

we obtain the kinetic equation for breather (soliton) gas.

Remark In the general 2D (spectral) case we replace

$$\int_{\Gamma^+} \dots |d\mu| \to \iint_{\Lambda^+} \dots d\xi d\zeta$$

where $\mu = \xi + i\zeta$ and $\Lambda^+ \in \mathbb{C}^+$ is a 2D compact region.

Remarks

- Another kinetic equation is obtained for the carrier wave wavumber

$$\tilde{u}_t + \tilde{u}\tilde{s} = 0; \quad \tilde{u}(\eta, x, t) = \tilde{U}[u(\eta, x, t)], \quad \tilde{s} = \tilde{S}[u(\eta, x, t)]$$

- Velocity of a "trial" soliton/breather with $\eta \neq \Gamma$ propagating through a soliton (breather) gas with the density of states $u(\eta)$

$$s(\eta) = \frac{s_0(\eta) - \int_{\Gamma^+} \Delta(\eta, \mu) u(\mu) s(\mu) |d\mu|}{1 - \int_{\Gamma^+} \Delta(\eta, \mu) u(\mu) |d\mu|}.$$

- For a soliton with spectral parameter η propagating through the bound state soliton condensate with $\Gamma = [-iq, iq]$ we obtain:

$$s(\eta) = -\frac{4\Im\eta\Re\eta}{\Im\sqrt{\eta^2 + q^2}}.$$

— an experimentally verifiable quantity.

Some explicit solutions of the kinetic equation

Watch out for the talk by Thibault Congy tomorrow

Multi-component hydrodynamic reductions

Let

$$u(\eta, x, t) = \sum_{j=1}^{M} w^j(x,t)\delta(\eta - \eta^{(j)}),$$

Then the kinetic equation becomes a system of quasilinear conservation laws

$$(w^j)_t + (w^j s^j)_x = 0, \qquad j = 1, \dots, M$$

with closure conditions

$$s^j = s_0^j + \sum_{m=1, m\neq j}^{M} \Delta_{jm} w^m (s^j - s^m), \quad j = 1, 2, \dots M,$$

where $s^i(x,t) = s(\eta^{(j)}, x, t)$.

- Hyperbolic, linearly degenerate, integrable hydrodynamic type system (*El, Kamchatnov, Pavlov & Zykov, J. Nonlin. Sci 2011*)

Shock tube problem for breather/soliton gas

Consider the two-component reduction

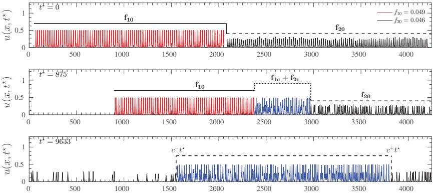
$$(w^j)_t + (w^j s^j)_x = 0, \qquad j = 1, 2.$$

$$s^1 = s^1_0 + \frac{\Delta_{12} w^2 (s^1_0 - s^2_0)}{1 - (\Delta_{12} w^2 + \Delta_{21} w^1)}, \quad s^2 = s^2_0 - \frac{\Delta_{21} w^1 (s^1_0 - s^2_0)}{1 - (\Delta_{12} w^2 + \Delta_{21} w^1)}.$$

with the "shock tube" initial conditions

$$\begin{cases} w^1(x,0) = w^1_0, \quad w^2(x,0) = 0\,, & x < 0, \\ w^2(x,0) = w^2_0, \quad w^1(x,0) = 0, & x > 0, \end{cases}$$

Assume $s^1_0 > s^2_0 > 0$



Numerical sumulations of the soliton gas shock tube problem (KdV) (*Carbone, El and Dutykh, EPL 2016*)

Shock tube problem: weak solution

The weak solution for w^1 and w^2 has a piecewise constant form:

$$
w^1(x,t) = \begin{cases} w_0^1, & x < c^- t, \\ w_c^1, & c^- t < x < c^+ t, \\ 0, & x > c^+ t. \end{cases}
$$

$$
w^2(x,t) = \begin{cases} 0, & x < c^- t, \\ w_c^2, & c^- t < x < c^+ t, \\ w_0^2, & x > c^+ t. \end{cases} \tag{1}
$$

where

$$
w_c^1 = \frac{w_0^1(1 - \Delta_{21} w_0^2)}{1 - \Delta_{12}\Delta_{21} w_0^1 w_0^2}, \quad w_c^2 = \frac{w_0^2(1 - \Delta_{12} w_0^1)}{1 - \Delta_{12}\Delta_{21} w_0^1 w_0^2},
$$

$$
c^- = s_0^2 - \frac{(s_0^1 - s_0^2)\Delta_{12} w_c^1}{1 - (\Delta_{12} w_c^1 + \Delta_{21} w_c^2)}, \quad c^+ = s_0^1 + \frac{(s_0^1 - s_0^2)\Delta_{21} w_c^2}{1 - (\Delta_{12} w_c^1 + \Delta_{21} w_c^2)}.
$$

Conclusions

- **Nonlinear dispersion relations and kinetic equations are derived for soliton and breather gases of the focusing NLS equation**;
- **The spectral scaling plays crucial role in the balance of terms in the nolinear dispersion relations**
- **Sub-exponential scaling corresponds to a soliton/breather condensate**

Thank you
for your attention!