Wave turbulence in a Bose–Einstein condensate: thermalisation and out-of-equilibrium steady states

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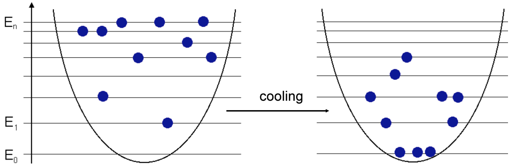
[PRA 2009; Physica D 2012; PRA 2014]

Outline

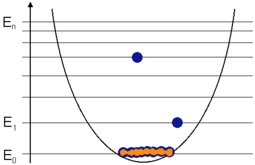
- Theoretical background:
  Bose–Einstein condensates (BECs),
  Gross–Pitaevskii model,
  Wave turbulence kinetic equation.
- Considering the 3D case:
  Condensation process,
  WT direct cascade,
  Critical Balance,
  Bogoliubov turbulence.
- Considering the 2D case:
  no BEC in infinite system,
  Berezinsky–Kosterlitz–Thouless transition,
  quantum vortex dynamics.

What is a Bose-Einstein condensate?



Boson system in a confining potential.

- Bosons can occupy the same quantum state
- $E = \sum_k n_k E_k$
- $T \sim \langle E \rangle$

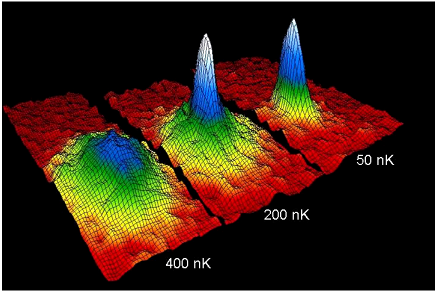


Bose-Einstein condensate and fluctuations in a confining potential.

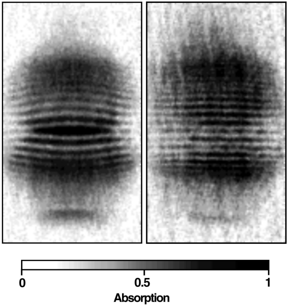
- A macroscopic fraction of particles occupies the lowest energy level
- Particle wave-functions overlap each other and quantum effects become macroscopic

E.A. Cornell, C.E. Wieman [JILA] and W. Ketterle [MIT]

Nobel Prize in Physics 2001 *"for the achievement of Bose-Einstein condensation in dilute gases of alkali atoms, and for early fundamental studies of the properties of the condensates"*.



Velocity distribution of a gas of rubidium during condensation [JILA group].

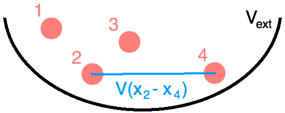


Interference between two BEC clouds [Ketterle et al.].

The Gross-Pitaevskii equation model

The many body hamiltonian operator is

$$\hat{H} = \int \hat{\Psi}^\dagger(\mathbf{x}_1)\left[\frac{\hat{p}^2}{2m} + V_{ext}(\mathbf{x}_1)\right]\hat{\Psi}(\mathbf{x}_1)d\mathbf{x}_1$$
$$+ \frac{1}{2}\int \hat{\Psi}^\dagger(\mathbf{x}_1)\hat{\Psi}^\dagger(\mathbf{x}_2)V(\mathbf{x}_1-\mathbf{x}_2)\hat{\Psi}(\mathbf{x}_2)\hat{\Psi}(\mathbf{x}_1)d\mathbf{x}_{12}$$
$$+ \dots$$



classical analogue

$$H = \sum_i \left[\frac{p_i^2}{2m} + V_{ext}(x_i)\right] + \frac{1}{2}\sum_{i,j} V(x_i - x_j).$$

If the system is cold and highly occupied

- $\hat{\Psi}(\mathbf{x}) = \psi(\mathbf{x}) + \delta\hat{\Psi}(\mathbf{x})$
- $V(\mathbf{x}_1 - \mathbf{x}_2) = \frac{4\pi\hbar^2 a}{m}\delta(\mathbf{x}_1 - \mathbf{x}_2)$

$i\hbar\partial_t\hat{\Psi} = [\hat{\Psi}, \hat{H}]$ leads to Gross-Pitaevskii equation (GPE)

$$i\hbar\partial_t\psi(\mathbf{x},t) = \left(-\frac{\hbar^2\nabla^2}{2m} + V_{ext}(\mathbf{x}) + \frac{4\pi\hbar^2 a}{m}|\psi(\mathbf{x},t)|^2\right)\psi(\mathbf{x},t)$$

$$i\hbar\frac{\partial\psi}{\partial t}+\frac{\hbar^2}{2m}\nabla^2\psi-g|\psi|^2\psi=V\psi,\quad \text{with}\quad g=\frac{4\pi\hbar^2 a_s}{m}$$

Setting $V\equiv 0,\quad \psi\to\sqrt{\rho_\infty}\,\psi,\quad t\to\frac{\hbar}{g\rho_\infty}\,t,\quad x\to\xi\,x$

$$i\partial_t\psi+\frac{1}{2}\left(\frac{\sqrt{2}\,\hbar^2}{mg\rho_\infty}\,\xi^{-2}\right)\nabla^2\psi-\frac{1}{2}|\psi|^2\psi=0\quad\Longrightarrow\quad\xi=\frac{\sqrt[4]{2}\,\hbar}{\sqrt{mg\rho_\infty}}$$

At scales $>\xi$ the nonlinear term dominates (phonons), at scales $<\xi$ the linear (kinetic) term becomes more important (free-particle excitations).

$$i\partial_t\psi+\beta\nabla^2\psi-\alpha|\psi|^2\psi=0$$

^{4}He	Alkali BECs
$\xi\sim\mathring{A},\quad L/\xi\simeq 10^4-10^5$	$\xi(m,a_s,\rho_\infty),\quad L/\xi\simeq 1-10^2$
Only qualitative model for liquid helium!	Very good model when $T\simeq 0$

$$i\frac{\partial\psi}{\partial t} + \nabla^2\psi - |\psi|^2\psi = 0, \qquad \begin{cases} M = \int |\psi|^2 d\mathbf{x} = \int \rho d\mathbf{x} \\ H = \int |\nabla\psi|^2 - \frac{1}{2}|\psi|^4 d\mathbf{x} \end{cases}$$

In general two conserved quantities M and H but in one-dimensional physical space the equation is integrable, infinite conserved quantities!

Madelung's transformation $\boxed{\psi(\mathbf{x},t) = \sqrt{\rho(\mathbf{x},t)}e^{i\theta(\mathbf{x},t)}, \quad \mathbf{v} = 2\nabla\theta}$

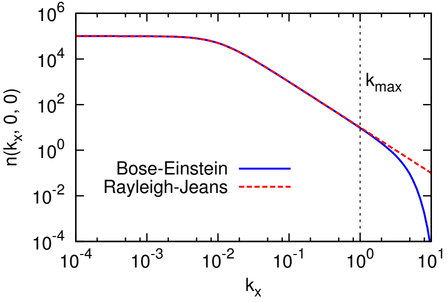
$$\begin{cases} \frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0 \\ \rho\left(\frac{\partial v_j}{\partial t} + v_k\frac{\partial v_j}{\partial x_k}\right) = -\frac{\partial p}{\partial x_j} + \frac{\partial\Sigma_{jk}}{\partial x_k} \end{cases}$$

- $p = \rho^2$ is a pressure term, $\Sigma_{jk} = \rho\frac{\partial^2(\ln\rho)}{\partial x_j\partial x_k}$ is the quatum stress tensor
- GPE describes an inviscid, irrotational, barotropic fluid.

The non-dimensional GPE is a particular case of the nonlinear Schrödinger equation, very important model in many physical systems.

Equilibrium distributions for BECs

For a non-interacting boson system at rest having temperature T and chemical potential μ:



Distributions having $T = 10$ and $\mu = 10^{-4}$.

- Bose-Einstein statistics

$$n_{BE}(\mathbf{k}) = \frac{1}{e^{\frac{|\mathbf{k}|^2+\mu}{T}} - 1}$$

- Rayleigh-Jeans

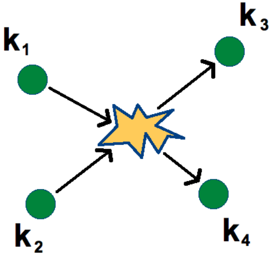
$$n_{RJ}(\mathbf{k}) = \frac{T}{|\mathbf{k}|^2 + \mu}$$

Bose-Einstein statistics reduces to the Rayleigh-Jeans distribution for $T \gg |\mathbf{k}|^2 + \mu$, we can define then a k_{max} of validity!

Kinetic equation and thermodynamic solution

Given $n_k \delta(\mathbf{k} - \mathbf{k}') = \langle \tilde{\psi}_k \tilde{\psi}^*_{k'} \rangle$, one finds

$$\frac{\partial n_1}{\partial t} = 4\pi \int n_1 n_2 n_3 n_4 \left(\frac{1}{n_1} + \frac{1}{n_2} - \frac{1}{n_3} - \frac{1}{n_4} \right) \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4)$$
$$\times \delta(\omega_1 + \omega_2 - \omega_3 - \omega_4) d\mathbf{k}_{234}, \quad \omega_i = |\mathbf{k}_i|^2$$



Elastic collision satisfying resonant conditions

- only resonant interactions

$$\begin{cases} \mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3 + \mathbf{k}_4 \\ \omega_1 + \omega_2 = \omega_3 + \omega_4 \end{cases}$$

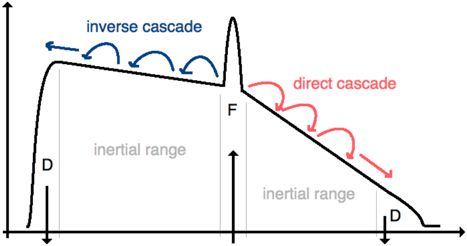
- $S(t) = \int \log n_{\mathbf{k}} d\mathbf{k}$, $\dot{S}(t) \geq 0$
- analogies with Boltzmann integral
- Rayleigh-Jeans thermodynamic

$$n(\mathbf{k}, t) = \frac{T}{\mu + \mathbf{a} \cdot \mathbf{k} + \omega(\mathbf{k})}$$

Steady cascade solutions [D.P., Nazarenko, Onorato, PRA 2009]

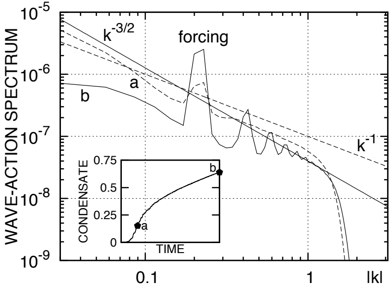
$$i\frac{\partial\psi}{\partial t} + \nabla^2\psi - |\psi|^2\psi = \mathcal{F} + \mathcal{D}$$

Supposing statistical isotropy in physical space $n(\mathbf{k}, t) = n(k, t)$
Kolmogorov-Zakharov solutions of kinetic equation: constant flux of energy (direct cascade) and particles (inverse cascade) in GPE



- 2 conserved quantities, 2 cascades
- direct energy cascade $n_{1D}(k) \sim k^{-1}$
- inverse cascade with $n_{1D}(k) \sim k^{-1/3}$

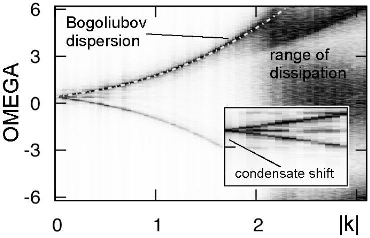
Condensate growth [D.P., Nazarenko, Onorato, PRA 2009]



$n_{1D}(k)$ at different steps.

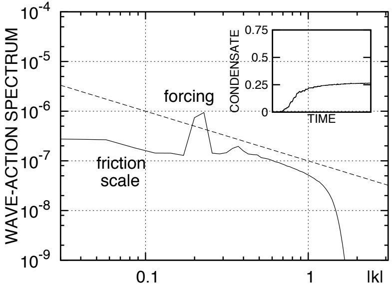
- forcing at large scales
- hyper-viscosity at small scales

- strong condensate
  $c_0 = |\tilde{\psi}(\mathbf{k}=0)| \gg |\tilde{\psi}(\mathbf{k}\neq 0)|$
- condensate growth alters the WWT dynamics



Dispersion relation. Bogoliubov is $\omega(k) = c_0^2 \pm k\sqrt{2c_0^2 + k^2}$.

The WT regime [D.P., Nazarenko, Onorato, PRA 2009]

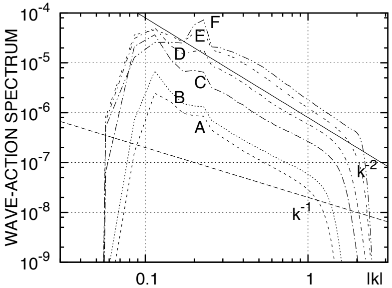


$n_{1D}(k)$ spectrum at final stage. The dashed line is the WWT prediction.

- forcing at large scales
- hyper-viscosity dissipation at small scales
- friction at scales larger than forcing to arrest the inverse cascade
- condensate growth is stopped

Steady regime which agrees with WT direct energy cascade prediction.

The critical balance regime [D.P., Nazarenko, Onorato, PRA 2009]



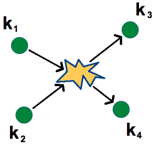
$n_{1D}(k)$ at final stage for various forcing coefficient. Black line is a k^{-2} slope.

- hypo-viscosity at large scales to arrest the inverse cascade
- suppression of the condensate fraction
- wide range of forcing coefficient: from $f_0 = 0.05$ (A) to $f_0 = 3$ (F)

A scale-by-scale energy balance between H_{NL} and H_{Lin} in Fourier space can explain $n_{1D} \sim k^{-2}$

Weak wave turbulence for $i\partial_t\psi + \nabla^2\psi - |\psi|^2\psi = 0$

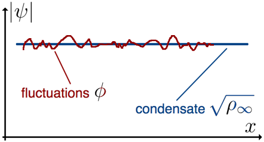
- Wave turbulence regime,
  small nonlinearity $\left||\psi|^2\psi\right| \ll \left|\nabla^2\psi\right|$
  4-wave interaction resonance processes



$$\frac{\partial n_1}{\partial t} = \int n_1 n_2 n_3 n_4 \left(\frac{1}{n_1} + \frac{1}{n_2} - \frac{1}{n_3} - \frac{1}{n_4}\right) \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4)$$
$$\times\delta(\omega_1 + \omega_2 - \omega_3 - \omega_4) d\mathbf{k}_{234}, \quad \omega_i = |\mathbf{k}_i|^2, \quad n_k \sim |\psi_k|^2$$

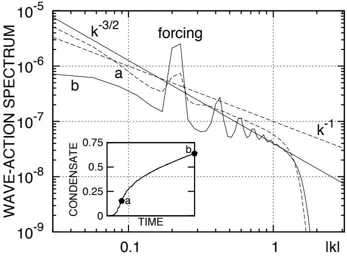
equilibrium steady state $\Longrightarrow$ Rayleigh-Jeans $n(\mathbf{k}) = \frac{T}{\omega(\mathbf{k})+\mu}$

- Strong condensate regime
  $\psi(\mathbf{x}, t) = \rho_0(t) + \phi(\mathbf{x}, t)$, $|\rho_0| \gg |\phi|$
  3-wave phonons interaction processes



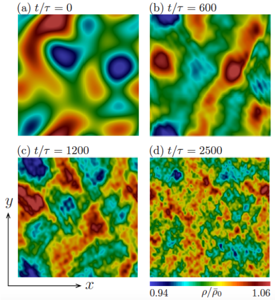
$$\omega_{Bog}(\mathbf{k}) = |\mathbf{k}|\sqrt{2\rho_0 + |\mathbf{k}|^2} \quad \Longrightarrow \quad |b(\mathbf{k})|^2 = \frac{T}{\omega_{Bog}(\mathbf{k})}$$

Bogoliubov wave turbulence

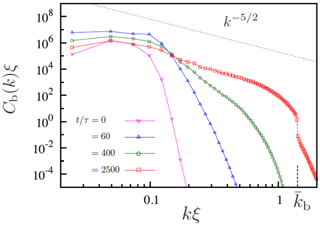


[D.P., Nazarenko, Onorato, PRA 2009]

- $n_{1D}(k) \propto k^{-3/2}$ on the hypothesis a_k and $a^{\star}_{-k}$ independent
- however because the Bogoliubov modes b_k and $b^{\star}_{-k}$ are now independent, one obtains $n_{1D}(k) \propto k^{-7/2}$ as derived in [Fujimoto & Tsubota, PRA 2015]



[Fujimoto & Tsubota, PRA 2015]



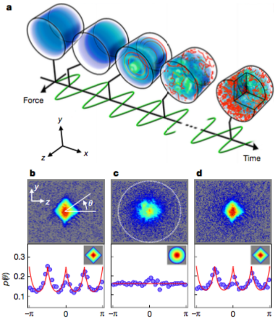
Recent BEC experiment [Navon et al., Nature 2016]

LETTER

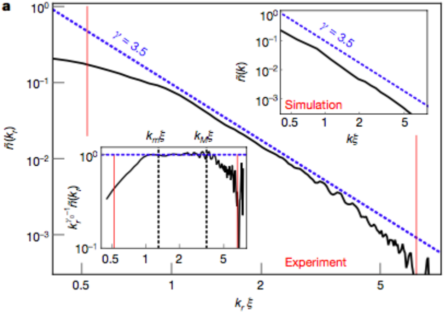
doi:10.1038/nature20114

Emergence of a turbulent cascade in a quantum gas

Nir Navon[1], Alexander L. Gaunt[1], Robert P. Smith[1] & Zoran Hadzibabic[1]



sketch of the experiment



reported spectra

No BEC in infinite 2D system! [Connaughton *et al.*, PRL 2005]

In the 3D case:

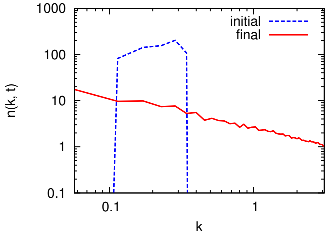
$$\frac{N}{V} = \int_0^{k_{max}} \frac{T}{k^2+\mu}\, 4\pi k^2\, dk$$
$$= 4\pi T \left[k_{max} - \sqrt{\mu}\, arctg\left(\frac{k_{max}}{\sqrt{\mu}}\right)\right]$$
$$\frac{E}{V} = \int_0^{k_{max}} \frac{T}{k^2+\mu}\, k^2\, 4\pi k^2\, dk$$
$$= 4\pi T \left[\frac{k_{max}^3}{3} + \mu^{\frac{3}{2}}\, arctg\left(\frac{k_{max}}{\sqrt{\mu}}\right) - k_{max}\mu\right]$$

In the 2D case:

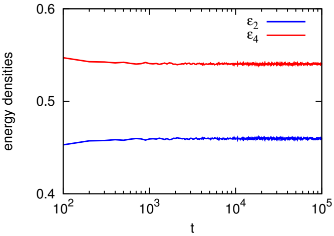
$$\frac{N}{V} = \int_0^{k_{max}} \frac{T}{k^2+\mu}\, 2\pi k\, dk$$
$$= \pi T \log\left(\frac{k_{max}^2+\mu}{\mu}\right)$$
$$\frac{E}{V} = \int_0^{k_{max}} \frac{T}{k^2+\mu}\, k^2\, 2\pi k\, dk$$
$$= \pi T \left[k_{max}^2 - \mu \log\left(\frac{k_{max}^2+\mu}{\mu}\right)\right]$$

- A measure of BEC is the correlation length $\lambda_c \sim 1/\sqrt{\mu}$
- In 3D, $\lambda_c = 0$ with a non-zero finite set (N, E) and $T_{BEC} \neq 0$.
- In 2D, $\lambda_c = 0 \Longrightarrow T_{BEC} = 0$ or divergent N!
- Rigorous proof using Mermin-Wagner theorem
- The first order correlation function $g_1(\mathbf{r}) = \langle \psi(\mathbf{x})\psi^*(\mathbf{x}+\mathbf{r})\rangle \sim e^{-|\mathbf{r}|}$

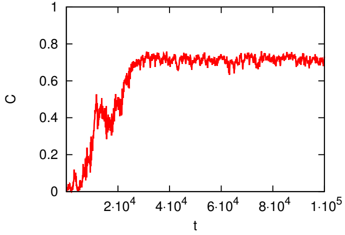
Thermalisation at $L = 256\,\xi$ [Nazarenko, Onorato and D.P., PRA 2014]



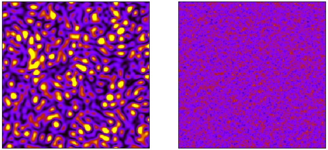
Evolution of the spectrum $n(k,t)$.



Evolution of the energy densities.

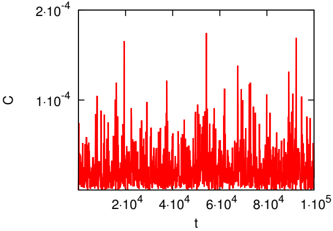


Evolution of the condensate $C(t)$.

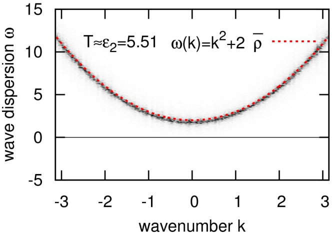


Snapshots corresponding to the initial and final density fields.

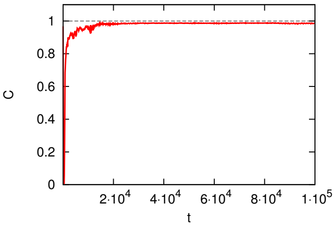
No condensate fraction:



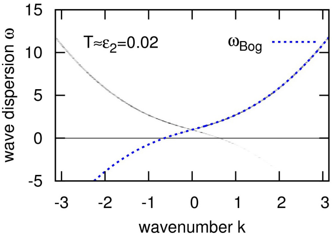
$$\omega(k) = k^2 + 2\bar{\rho}$$



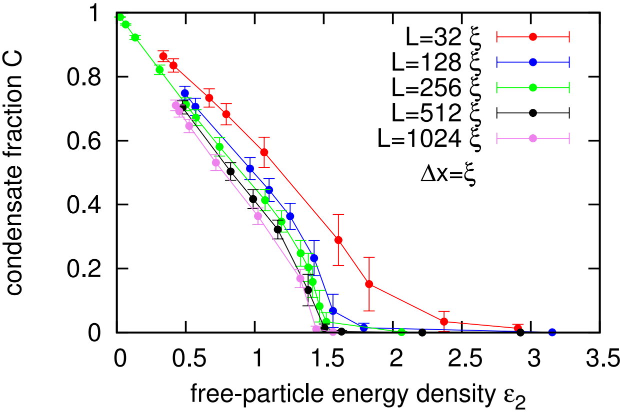
High condensate fraction:



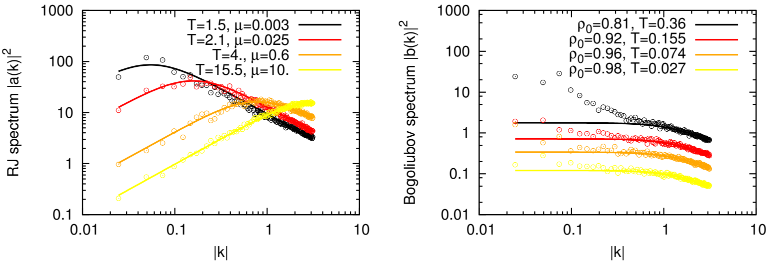
$$\omega_{Bog}(k) = k\sqrt{k^2 + 2\rho_0}$$



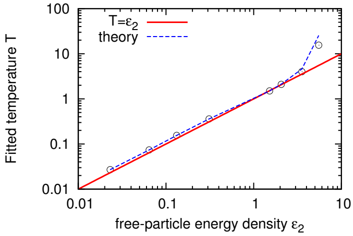
Condensate fraction measure [Nazarenko, Onorato and D.P., PRA 2014]



Condensate fraction measured in simulations having different box sizes L and different final steady linear energy densities $\epsilon_2 = \int |\nabla\psi|^2 dS/S$.



Estimation of the temperature in the two weakly nonlinear regimes by fitting with the predicted equilibrium distributions.



Estimated temperature with respect to the linear energy density ϵ_2.

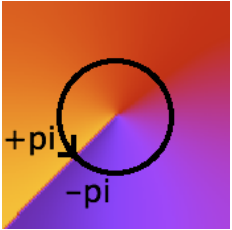
- for small temperatures $T \simeq \epsilon_2$
- integrating the RJ distribution

$$\epsilon_2 = \frac{T\, k_{max}^2}{\pi^2} - \mu\, \bar{\rho}$$

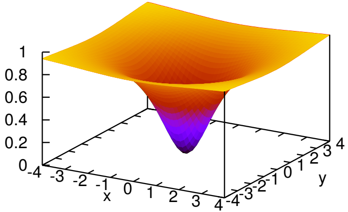
- no theoretical prediction for strong turbulent regime $\left| |\psi|^2 \psi \right| \simeq |\nabla^2 \psi|$

Two-dimensional quantum vortices

- using Madelung's transformation $\psi = \sqrt{\rho} e^{i\theta}$, $\mathbf{v} = 2\nabla\theta$
- a vortex is a hole in the density where phase changes of $\Delta\theta = 2\pi n, \;\; n \in \mathcal{N}$
- $\mathcal{C} = \oint \mathbf{v} \cdot d\mathbf{l} = 2 \oint \nabla\theta \cdot d\mathbf{l} = 2\Delta\theta$ is quantized. For the Stokes theorem if $\Delta\theta \neq 0$ the field ψ goes to zero at vortex core

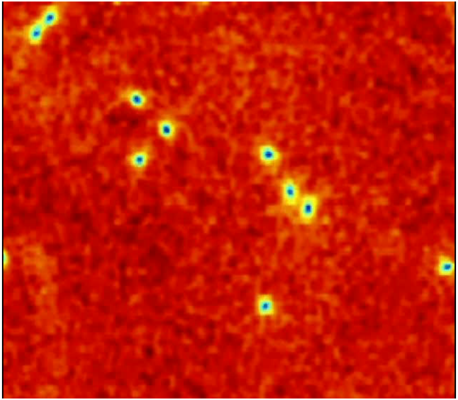


$\theta(\mathbf{x}, t)$ around a vortex.



$\rho(\mathbf{x}, t)$ around a vortex.

An example of 2D dynamics



Density field in a turbulent regime.

- chaotic quantum vortex dynamics
- nucleation and annihilation processes
- clustering
- sound emission

For the forced-dissipated 2D case and the role played by vortices refers to [Nazarenko & Onorato, Physica D 2006]

The Berezinsky-Kosterlitz-Thouless transition

$$E_v = 2\pi\,\rho_s \log\left(\frac{L}{\xi}\right)$$

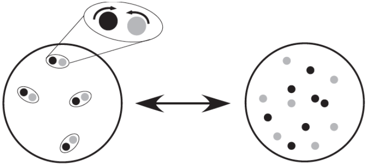
$$S = \log\left[\left(\frac{L}{\xi}\right)^2\right] = 2\log\left(\frac{L}{\xi}\right)$$

$$\Longrightarrow$$

$$F = E_v - TS = \frac{T}{2}\left(\rho_s\lambda^2 - 4\right)\log\left(\frac{L}{\xi}\right),$$

$\lambda = \sqrt{\frac{4\pi}{T}}$ is the thermal length

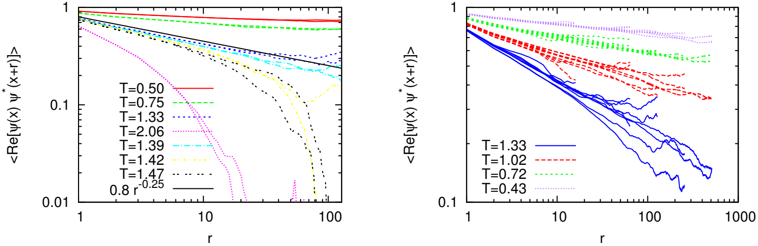
The free energy changes sign at temperature $T_{BKT} = \pi\rho_s$!



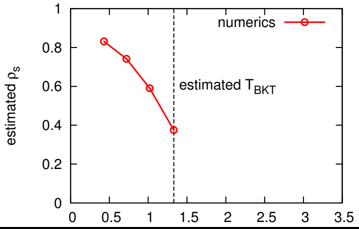
Schematic picture of BKT transition [Hadzibabic & Dalibard, Nuovo Cimento 2011].

- Above T_{BKT}, $F < 0$ so proliferation of new vortices is favourable
- Below T_{BKT}, $F > 0$ and vortices form dipoles
- Below T_{BKT}, first order correlation follows $g_1(r) = \rho_s\left(\frac{\xi}{r}\right)^{\alpha}$, $\quad \alpha = \frac{1}{\lambda^2\rho_s}$

The BKT transition temperature [Nazarenko, Onorato and D.P., PRA 2014]

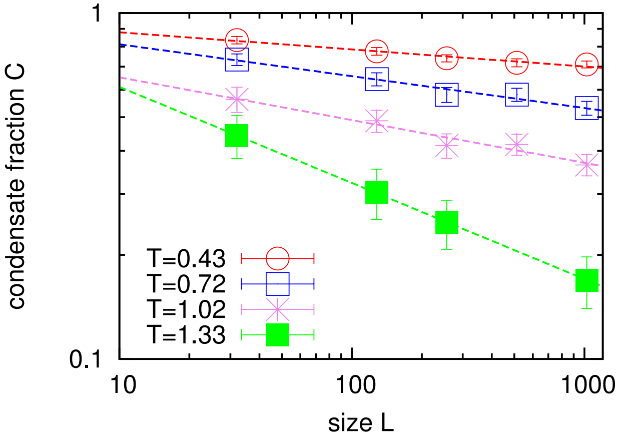


Correlation for different temperatures T and different system size L.



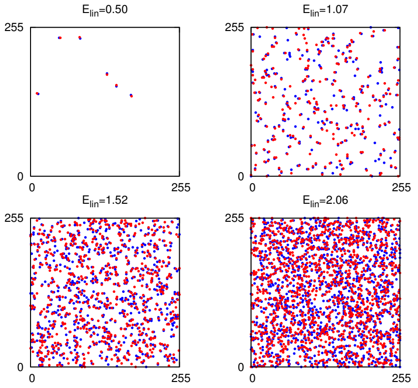
- the transition from exponential to power-law decay is around $T_{BKT} = 1.39$
- the exponent at T_{BKT} is exactly $\alpha = 1/4$
- transition independent of the system size!

Condensate fraction below T_{BKT}, $C = \frac{1}{\bar{\rho}L^2}\int g_1(\mathbf{r})d\mathbf{r} \simeq \frac{2\pi^{\alpha/2}\rho_s}{\bar{\rho}\,(2-\alpha)}\left(\frac{\xi}{L}\right)^{\alpha}$

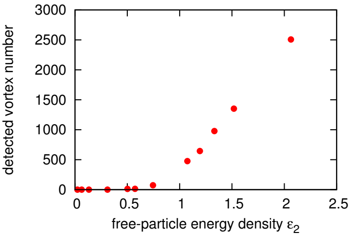


Condensate fraction measured in simulations having different box sizes L and different final steady linear energy densities ϵ_2.

The role of vortices [Nazarenko, Onorato and D.P., PRA 2014]



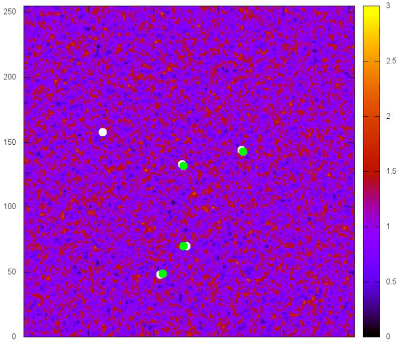
Evaluation of the quantum vortices at the final stage in systems having $L = 256$ and different temperatures T.



Number of vortices with respect to the linear energy density.

- clear predominance of dipole structures at $T < T_{BKT} = 1.39$
- smooth growth of vortex number increasing the temperature

Interesting vortex dynamics! [Nazarenko, Onorato and D.P., PRA 2014]



Evolution of the density field for a system with $L = 256$ and $T = 0.50$, well below T_{BKT}. Detected vortices are shown as green and white points depending on their orientation.

Conclusions

- In the 3D GP model, BEC spontaneously occurs in infinite system
- Two weakly nonlinear regimes exist, 4-wave (thermal, no condensed) regime and 3-wave (Bogoliubov, condensed) regime, where to observe KZ energy cascade spectra
- no BEC is possible in 2D infinite system, but (quasi-)condensation is recovered for finite systems
- BKT is the most important transition in 2D, driving also (quasi-)BEC!
- BKT seems to be size-independent (work in progress)
- vortices around T_{BKT} are not well defined hydrodynamic objects, intermittent creation and annihilation of dipoles

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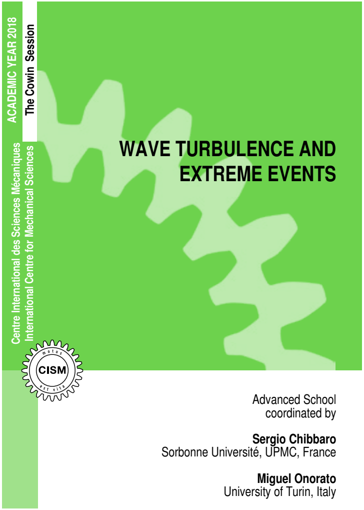

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Thanks for your attention!

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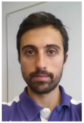

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