Wave turbulence in nonlinear optics and BEC

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In collaboration with

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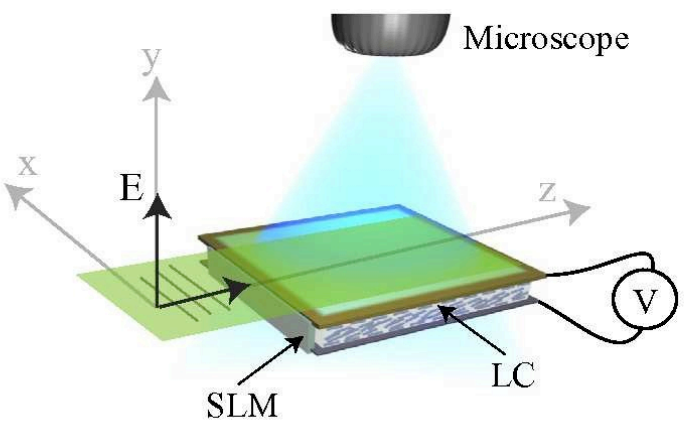
Theoretical Challenges in Wave Turbulence
Aston University, 8-9th December 2017

Optical WT

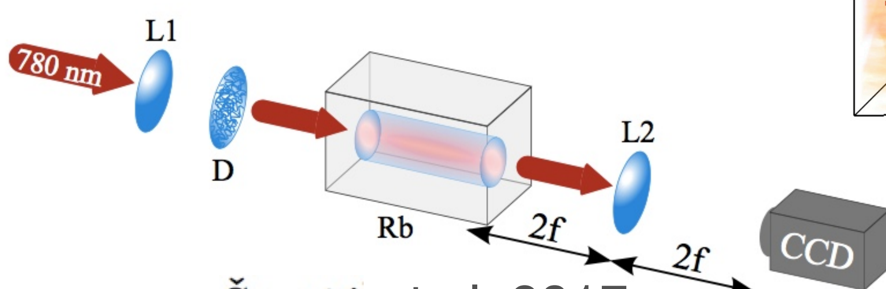
1D

Aston group



Bortolozzo et al, INPHYNI

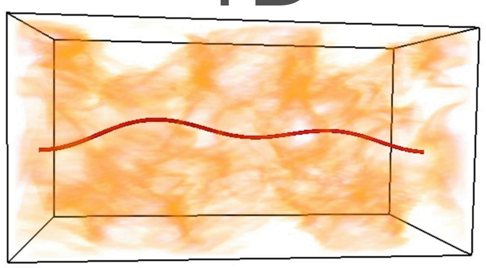
2D



Šantić et al, 2017
INPHYNI

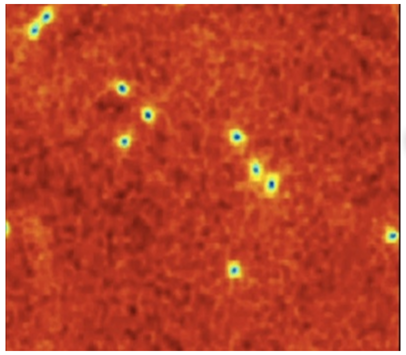
BEC WT

1D

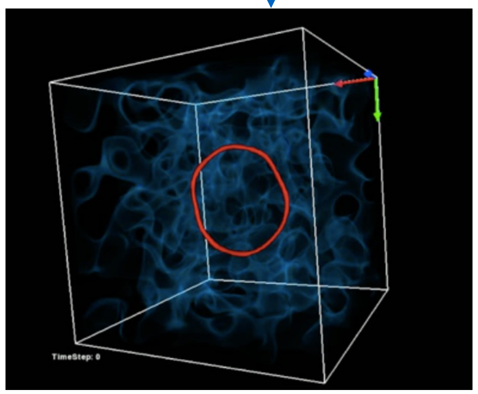


Kelvin waves

2D



3D



Davide's talk:
De-Broglie 4-wave turbulence (no condensate).
Bogolubov 3-wave turbulence.
Vortices.
Direct and inverse cascades.
Equilibrium states: condensates and quasi-condensates (BKT).

Liquid crystal and the Kelvin-wave systems

- Both systems are 1D and have similar dispersion relations, ω < k^2, which do not allow 4-wave (2 to 2) interactions.
- U(1) symmetry means that there are no odd-wave interactions.
- Hence similar theoretical techniques using

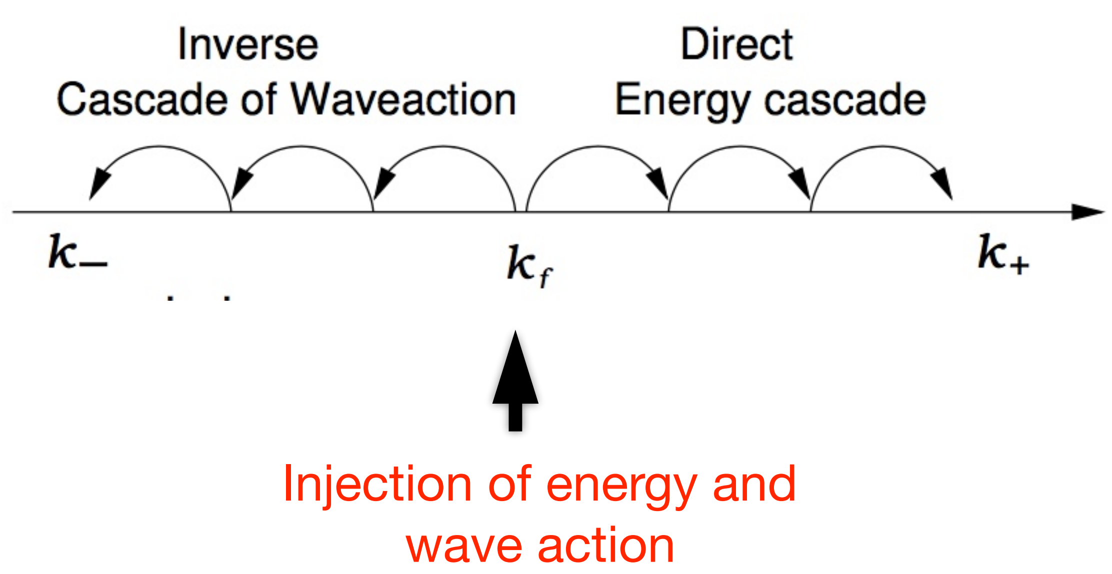
Optical Wave Turbulence

Properties

- Broadband spectrum
- Weak initial nonlinearities, random initial phases and amplitudes
- Turbulent cascades, photon condensation, modulational instability, solitons

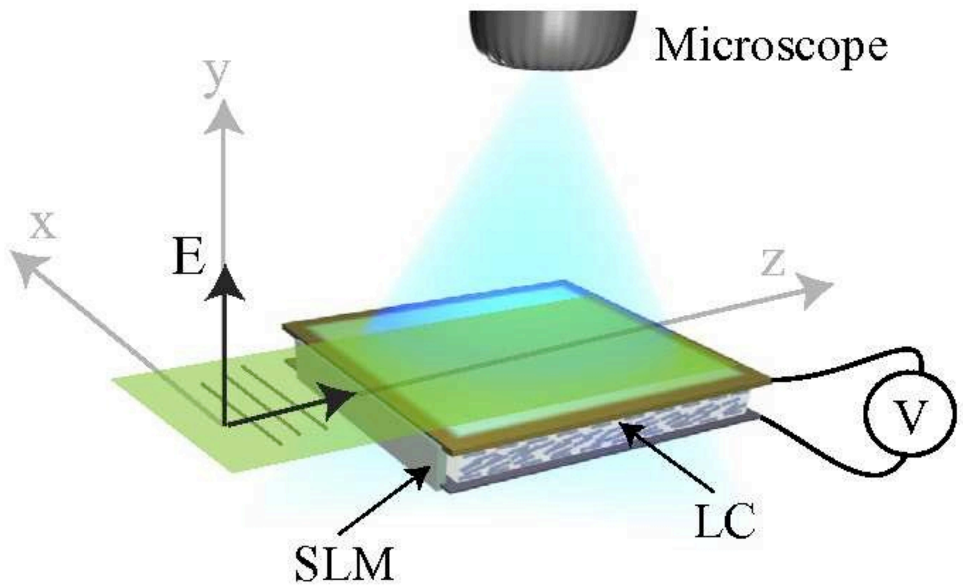
A dual cascade system

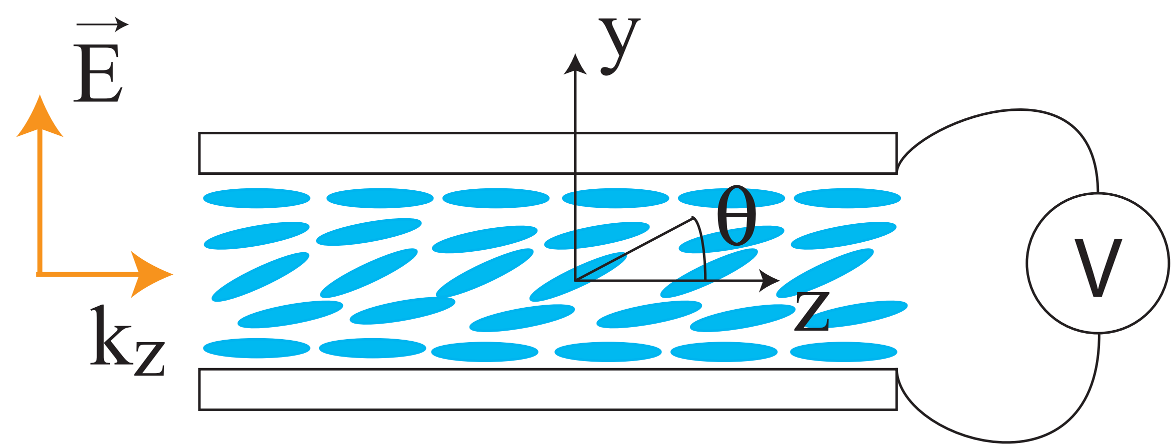
- Two sign-definite quadratic invariants: Energy and Wave action
- Fjørtoft argument leads to simultaneous cascades of both invariants



Experimental setup

Experiment by Bortolozzo and Residori at the INLN





- Propagate a light beam with randomised phases through a nematic liquid crystal cell
- Nonlinearity provided by the feedback loop:
- the refractive index depend on the molecules' orientation and
- The orientation dependence on the light intensity
- Measurements obtained from observing through the top of cell

Initial Condition

Requirements for an optical wave turbulent inverse cascade

- Low intensity optical beam for weak nonlinearity
- Beam modulated at high harmonics to ensure inertial range towards large scales
- Random phases
- Inverse cascade is finite capacity (initial condition will act as forcing reservoir)

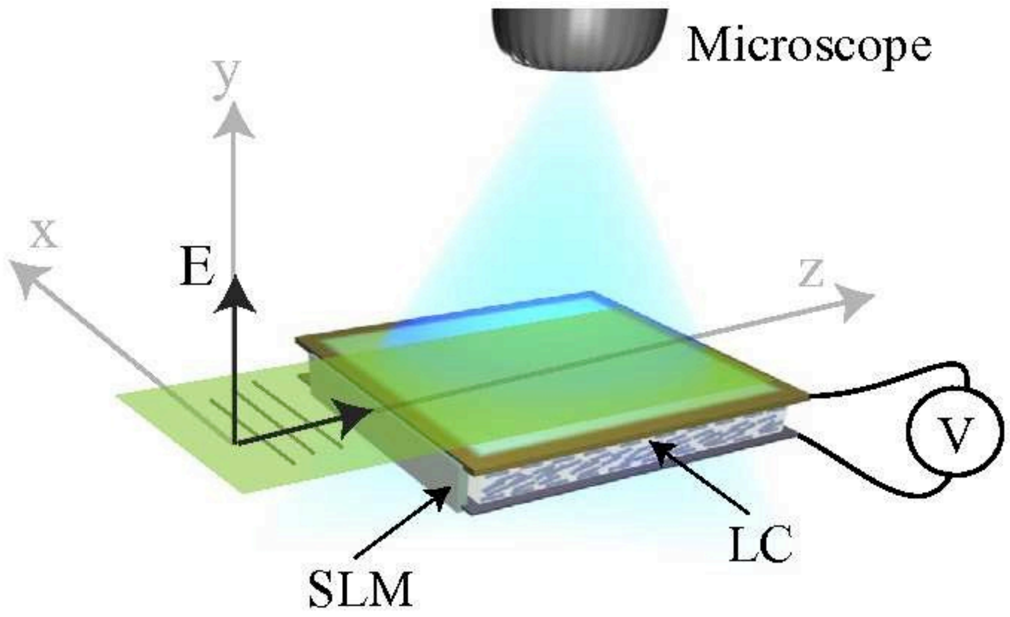
Experimental parameters

Experiment size $20\ \mathrm{mm} \times 30\ \mathrm{mm} \times 50\mu\mathrm{m}$

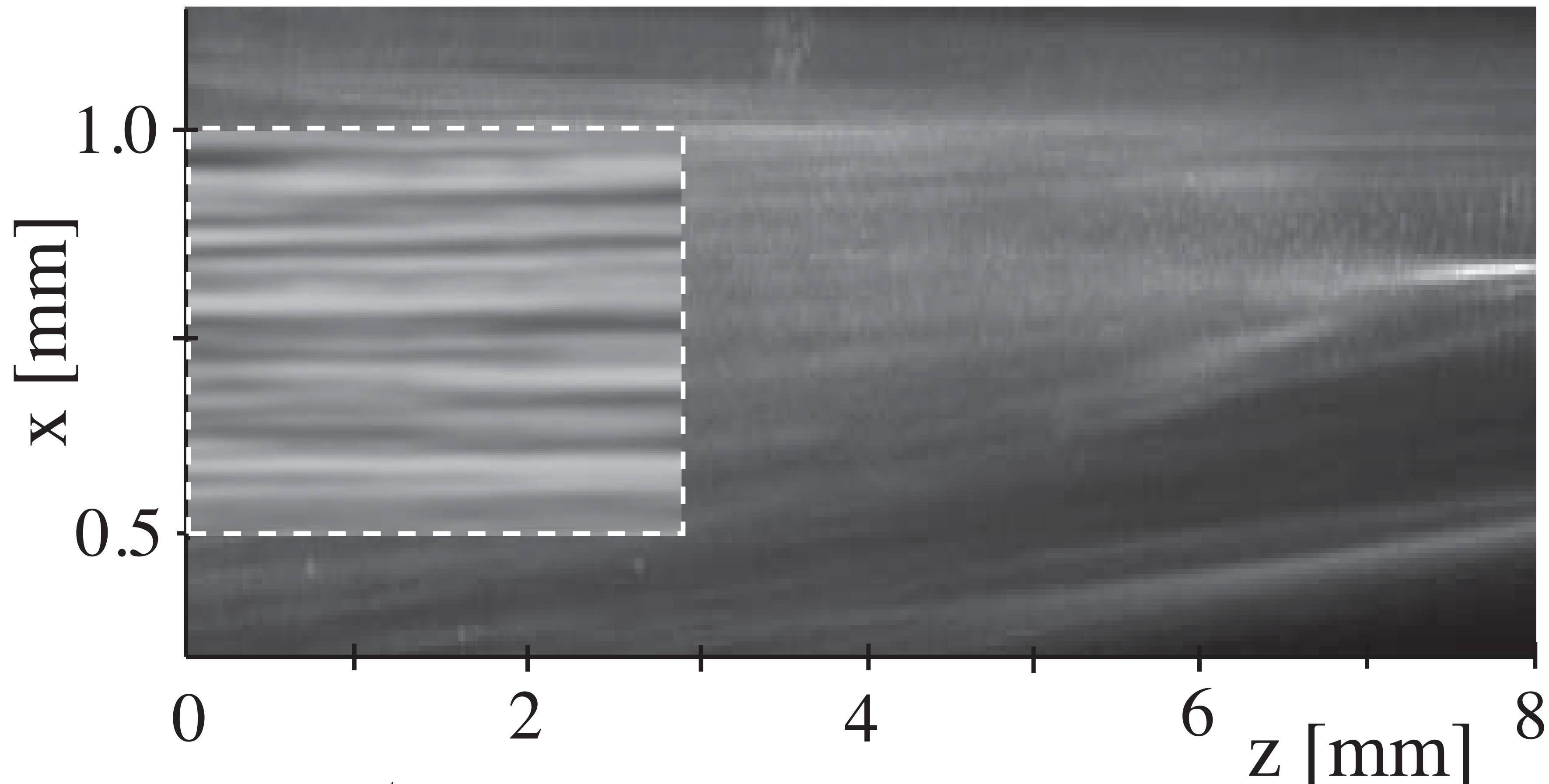
Forcing scale $l_f = \dfrac{2\pi}{k_f} \simeq 37 \pm 10\mu\mathrm{m}$

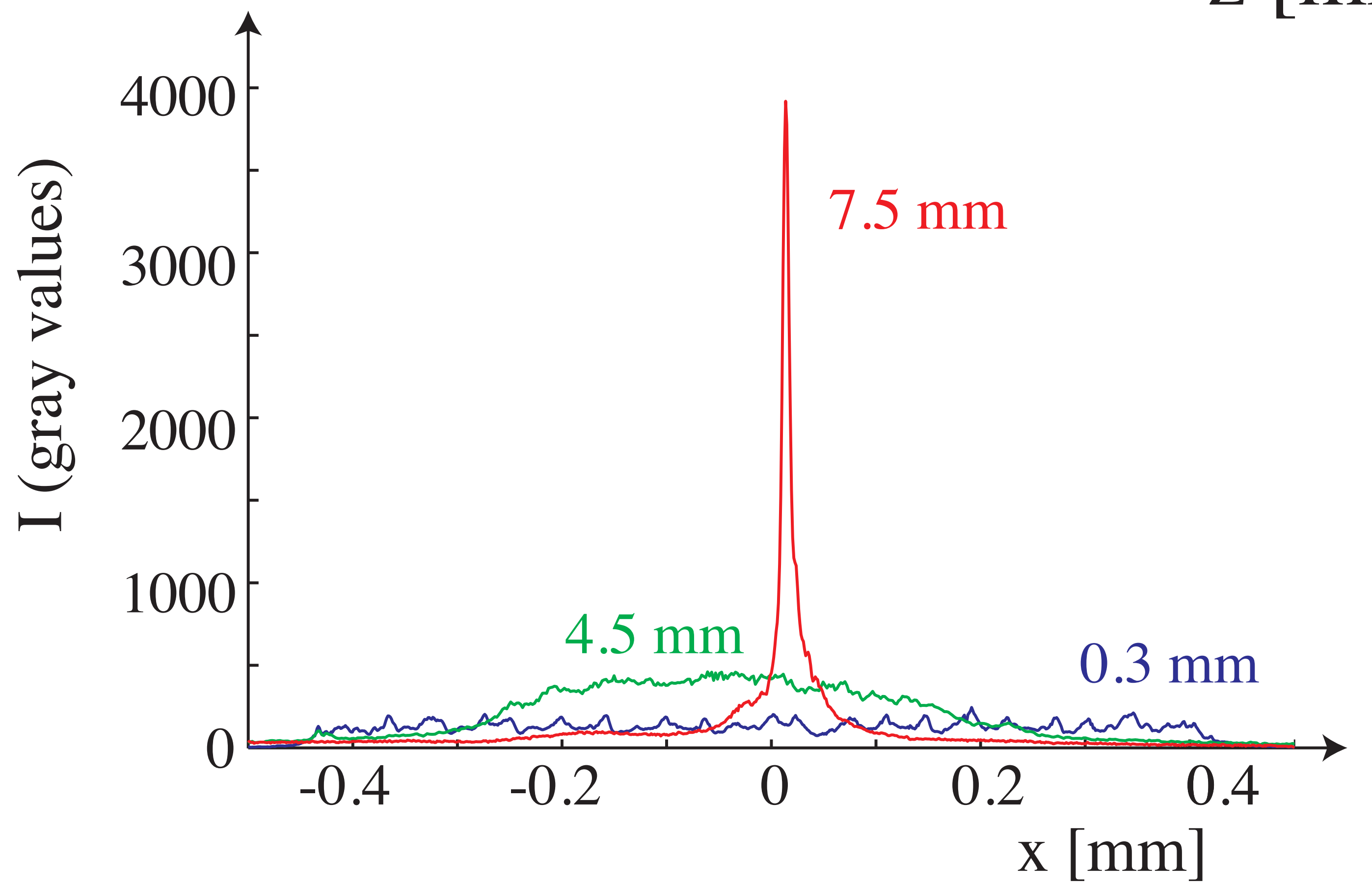
Dissipative scale $l_d = \dfrac{2\pi}{k_d} \simeq 10\mu\mathrm{m}$

Beam intensity $I \simeq 300\ \mathrm{mW/cm^2}$

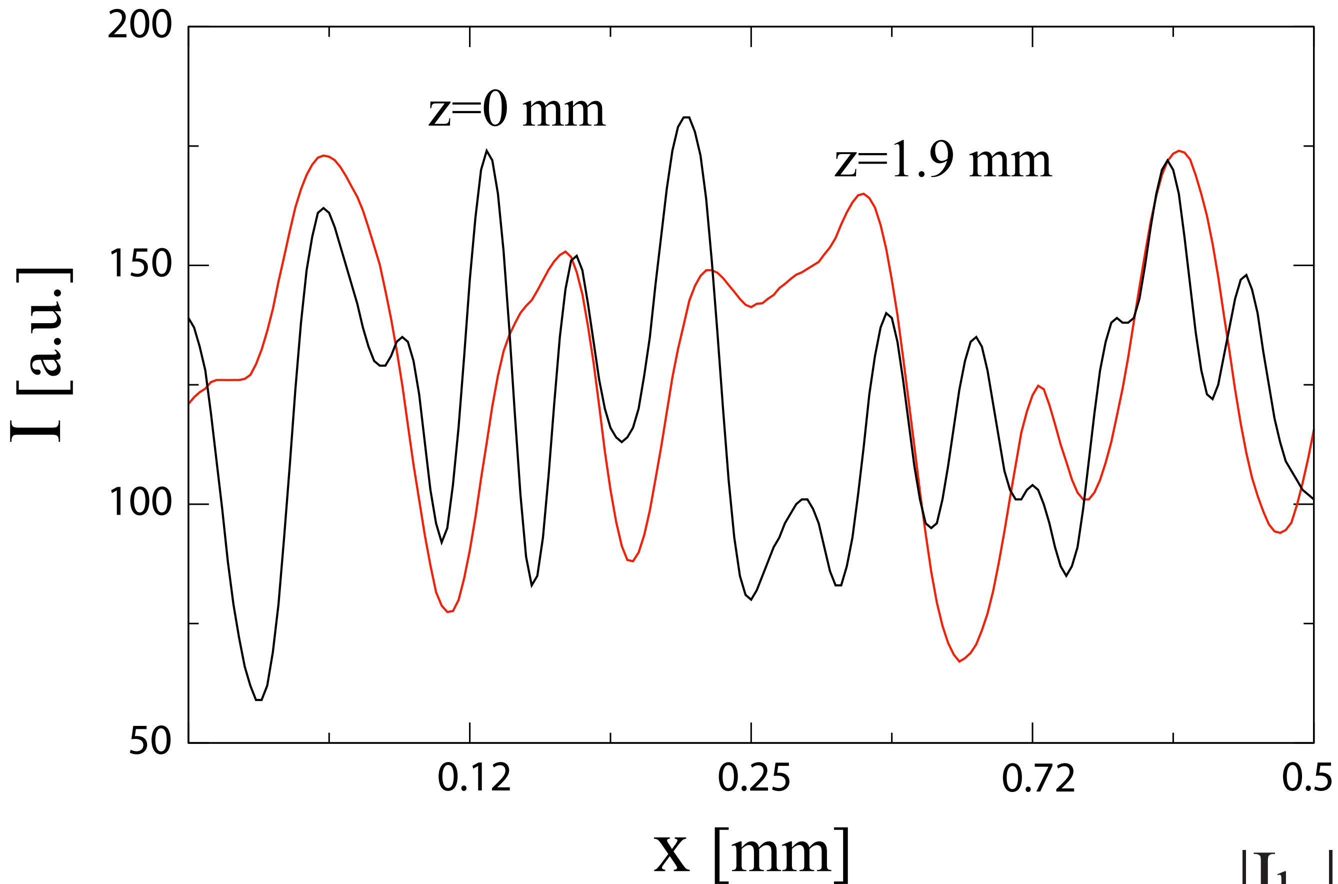


Photon Condensation and Solitons

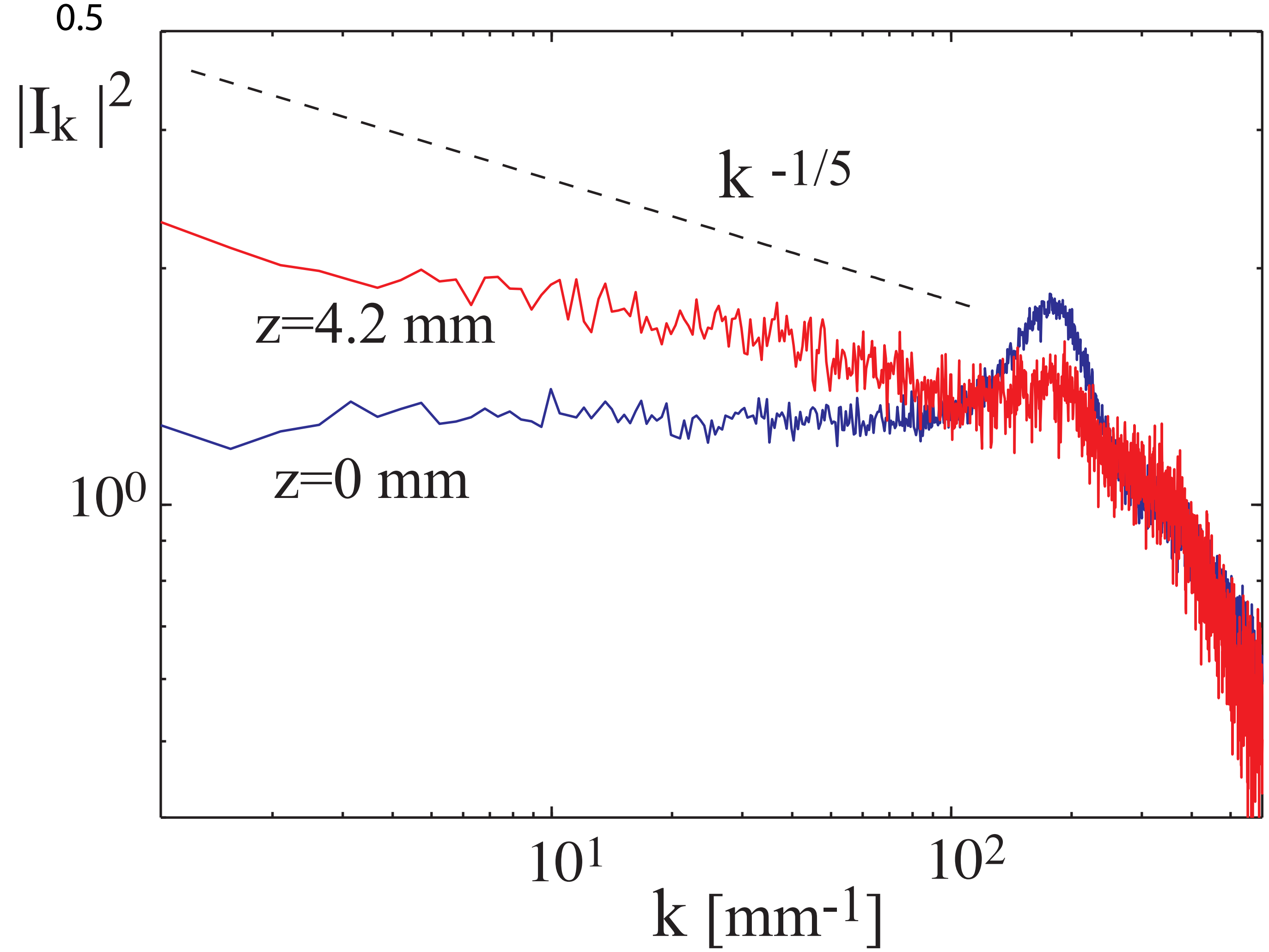




Inverse Cascade



Smoothing of the intensity field



Observation of an inverse cascade from measurement of the wave intensity spectrum

Theoretical Description

Equation of motion

$$2iq\frac{\partial\psi}{\partial z} = -\frac{\partial^2\psi}{\partial x^2} - \frac{\varepsilon_0 n_a^4 k_0^2 l_\xi^2}{4K}\psi\left(1 - l_\xi^2\frac{\partial^2}{\partial x^2}\right)^{-1}|\psi|^2$$

$$n_a^2 = n_e^2 - n_o^2, \qquad q^2 = k_0^2\left(n_o^2 + n_a^2/2\right), \qquad l_\xi = \left(\frac{\pi K}{2\Delta\varepsilon}\right)^{1/2}\frac{d}{V}$$

- l_ξ roughly dictates the dissipation scale of the experiment

Long-wave limit expansion $kl_\xi \ll 1$

$$2iq\frac{\partial\psi}{\partial z} = -\frac{\partial^2\psi}{\partial x^2} - \frac{\varepsilon_0 n_a^4 k_0^2 l_\xi^2}{4K}\left(\psi|\psi|^2 + l_\xi^2\psi\frac{\partial^2|\psi|^2}{\partial x^2}\right)$$

- At leading order we recover the 1D nonlinear Schrödinger equation
- For OWT we require non-integrability, so we expand up to subsequent order
- System is weakly non-integrable, expect dynamics close to NLS

Theory

Wave action representation of Hamiltonian

$$\psi = \sum_{\mathbf{k}} a_{\mathbf{k}}(z) \exp(i\,\mathbf{k}\,x)$$

$$\mathcal{H} = \sum_{\mathbf{k}} \omega_k a_{\mathbf{k}} a_{\mathbf{k}}^* + \sum_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4} T_{\mathbf{k}_3,\mathbf{k}_4}^{\mathbf{k}_1,\mathbf{k}_2}\, a_{\mathbf{k}_1} a_{\mathbf{k}_2} a_{\mathbf{k}_3}^* a_{\mathbf{k}_4}^*\, \delta\left(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4\right)$$

$\gg$

Weak nonlinearity

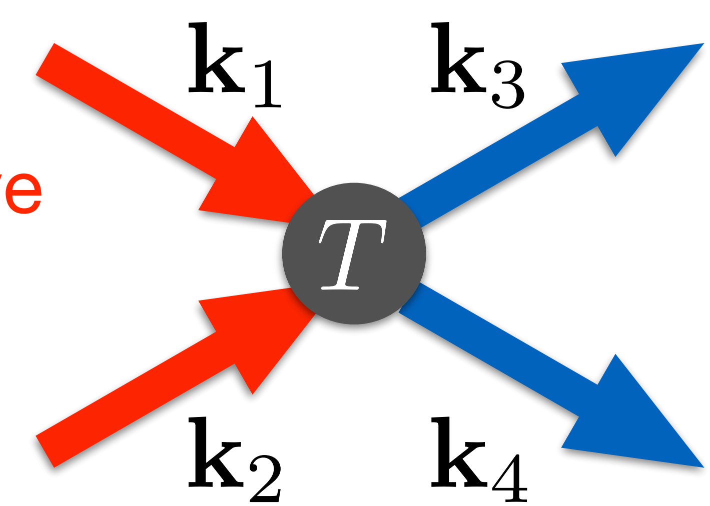
$$\omega_k = k^2$$

$$T_{\mathbf{k}_3,\mathbf{k}_4}^{\mathbf{k}_1,\mathbf{k}_2} = C + F(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4)$$

$$\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3 + \mathbf{k}_4$$

$$\omega_1 + \omega_2 = \omega_3 + \omega_4$$

Four-wave process



Absence of four-wave resonances

- Form of linear frequency prevents non-trivial solutions of resonance condition
- Four-wave turbulent mixing does not describe turbulent dynamics

Six-Wave Resonances

Canonical Transformation

- Remove the lowest order non-resonant nonlinear wave interactions whilst preserving linear dynamics using a change of variable

$$\mathcal{H} = \sum_{\mathbf{k}} \omega_k a_{\mathbf{k}} a_{\mathbf{k}}^* + \sum_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4} T_{\mathbf{k}_3,\mathbf{k}_4}^{\mathbf{k}_1,\mathbf{k}_2}\, a_{\mathbf{k}_1} a_{\mathbf{k}_2} a_{\mathbf{k}_3}^* a_{\mathbf{k}_4}^* \,\delta\left(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4\right)$$

Apply canonical transformation

$$a_{\mathbf{k}} = c_{\mathbf{k}} + o\left(c_{\mathbf{k}}^2\right)$$

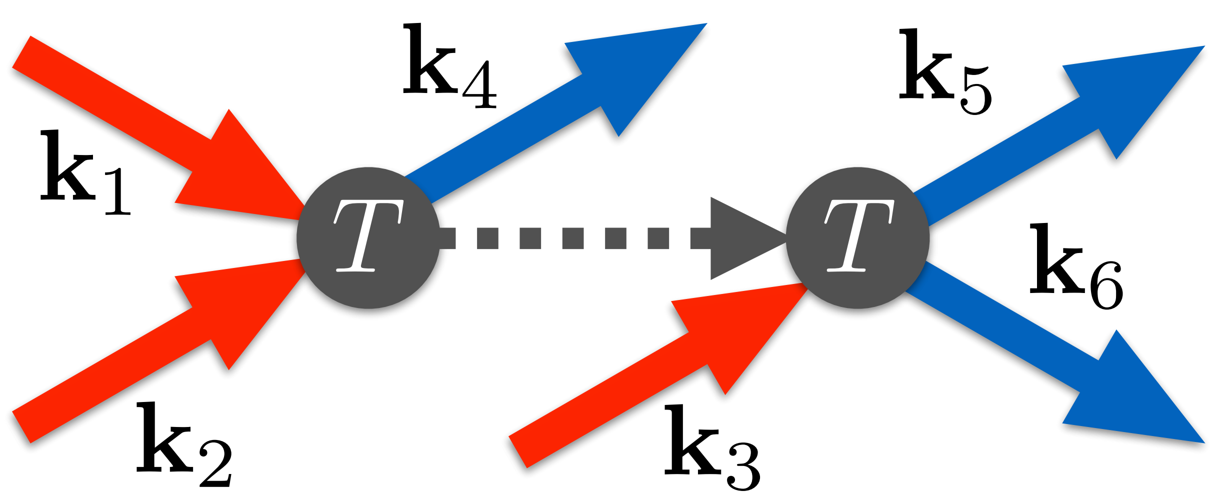
$$\mathcal{H} = \sum_{\mathbf{k}} \omega_k c_{\mathbf{k}} c_{\mathbf{k}}^* + \sum_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4,\mathbf{k}_5,\mathbf{k}_6} W_{\mathbf{k}_4,\mathbf{k}_5,\mathbf{k}_6}^{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3}\, c_{\mathbf{k}_1} c_{\mathbf{k}_2} c_{\mathbf{k}_3} c_{\mathbf{k}_4}^* c_{\mathbf{k}_5}^* c_{\mathbf{k}_6}^* \,\delta\left(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 - \mathbf{k}_4 - \mathbf{k}_5 - \mathbf{k}_6\right)$$

Resonant six-wave processes $W \sim T \circ T$

- Lowest nonlinear wave interaction becomes a resonant six-wave process

$$\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 = \mathbf{k}_4 + \mathbf{k}_5 + \mathbf{k}_6$$

$$\omega_1 + \omega_2 + \omega_3 = \omega_4 + \omega_5 + \omega_6$$



Kinetic Equation and Turbulent Spectra

Kinetic wave equation

- Evolution of a single harmonic is $2iq\dfrac{\partial c_{\mathbf{k}}}{\partial z} = \dfrac{\delta \mathcal{H}}{\partial c^*_{\mathbf{k}}}$
- WT theory leads to a kinetic equation for $n_{\mathbf{k}} = \langle c_{\mathbf{k}} c^*_{\mathbf{k}} \rangle$

$$\frac{\partial n_{\mathbf{k}}}{\partial z} = \frac{\pi}{6} \int \left| W^{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3}_{\mathbf{k}_4,\mathbf{k}_5,\mathbf{k}_6} \right|^2 n_{\mathbf{k}} n_2 n_3 n_4 n_5 n_6 \left[\frac{1}{n_{\mathbf{k}}} + \frac{1}{n_2} + \frac{1}{n_3} - \frac{1}{n_4} - \frac{1}{n_5} - \frac{1}{n_6} \right]$$
$$\times\, \delta\left(\mathbf{k} + \mathbf{k}_2 + \mathbf{k}_3 - \mathbf{k}_4 - \mathbf{k}_5 - \mathbf{k}_6\right) \delta\left(\omega_{\mathbf{k}} + \omega_2 + \omega_3 - \omega_4 - \omega_5 - \omega_6\right)\, d\mathbf{k}_2 d\mathbf{k}_3 d\mathbf{k}_4 d\mathbf{k}_5 d\mathbf{k}_6$$

Stationary equilibrium solutions

P energy flux, Q wave action flux

$P = 0, Q = 0 \;\Rightarrow\; n_{\mathbf{k}} = \dfrac{T}{\omega_k + \mu}$ — Thermodynamic equilibrium

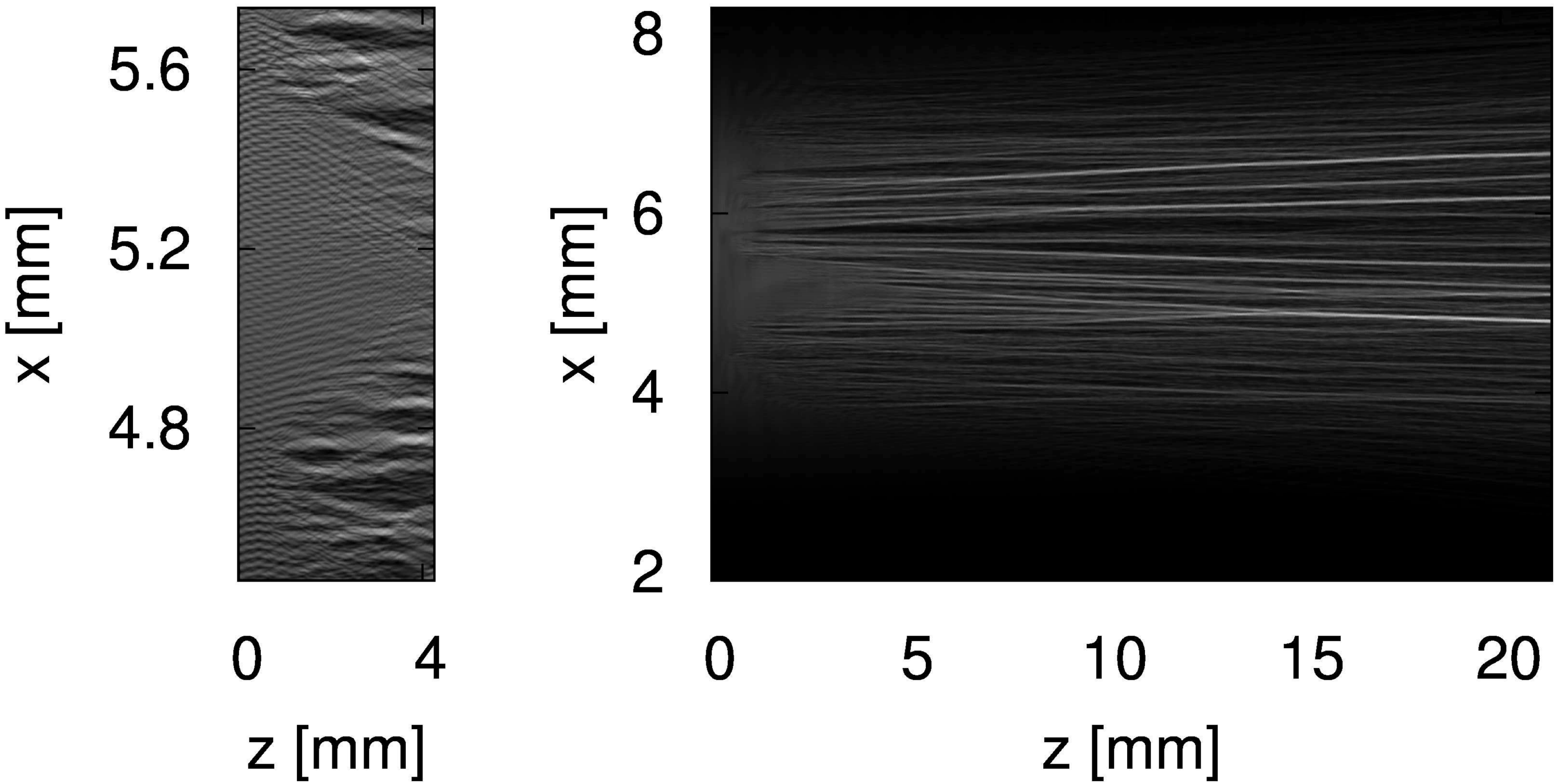
Stationary non-equilibrium (Kolmogorov-Zakharov) solutions

$P > 0, Q = 0 \;\Rightarrow\; n_{\mathbf{k}} \propto k^{-1}$ — Direct energy spectrum

$P = 0, Q < 0 \;\Rightarrow\; n_{\mathbf{k}} \propto k^{-3/5}$ — Inverse wave action spectrum (unrealisable due to wrong flux direction)

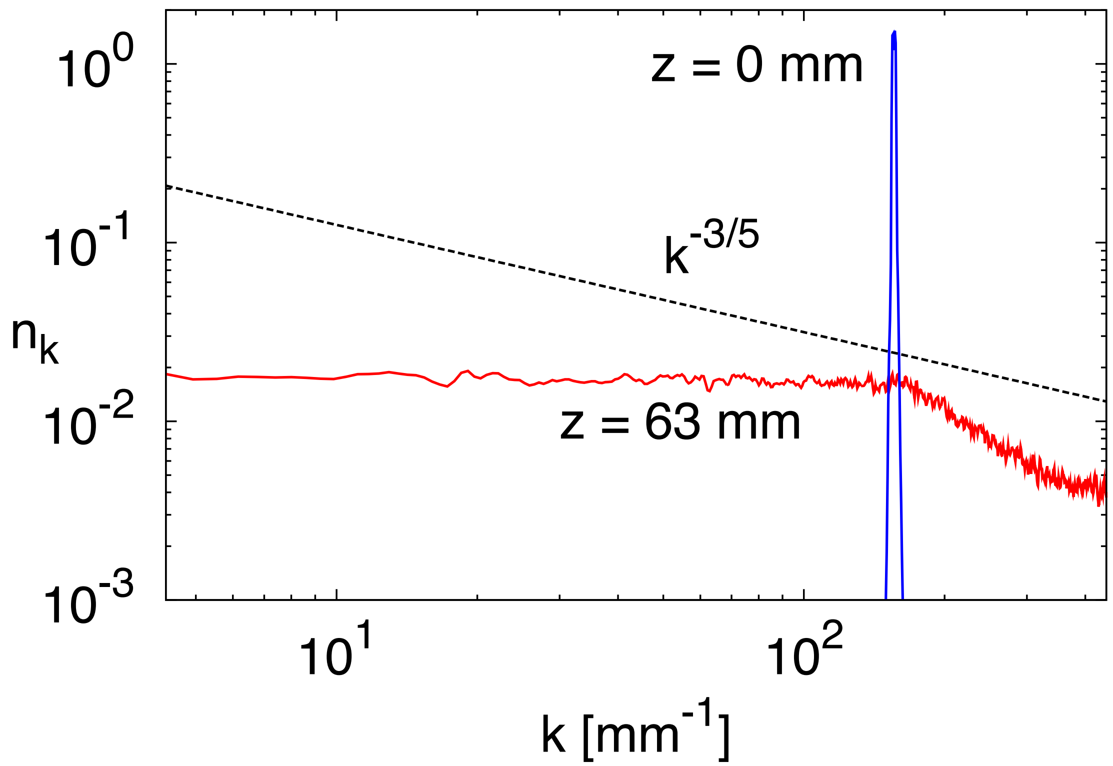
Numerical Simulation

Evolution of light intensity $I = |\psi|^2$



Numerical Simulation

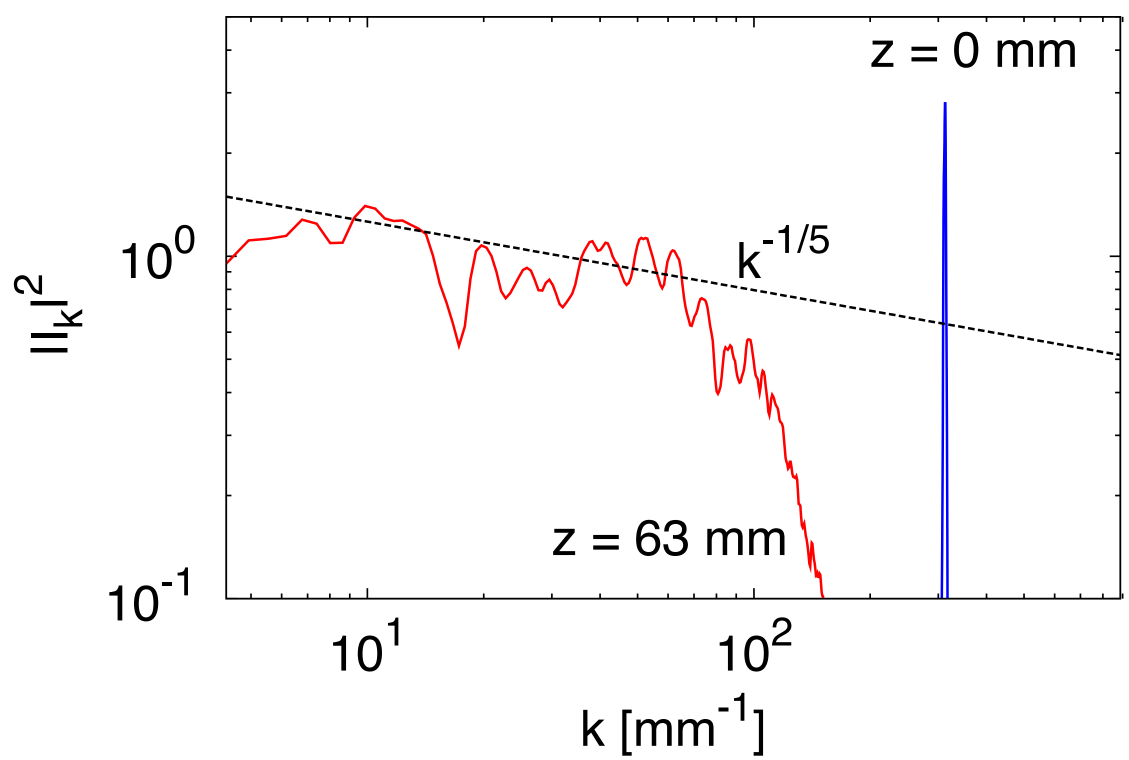
Wave action spectrum

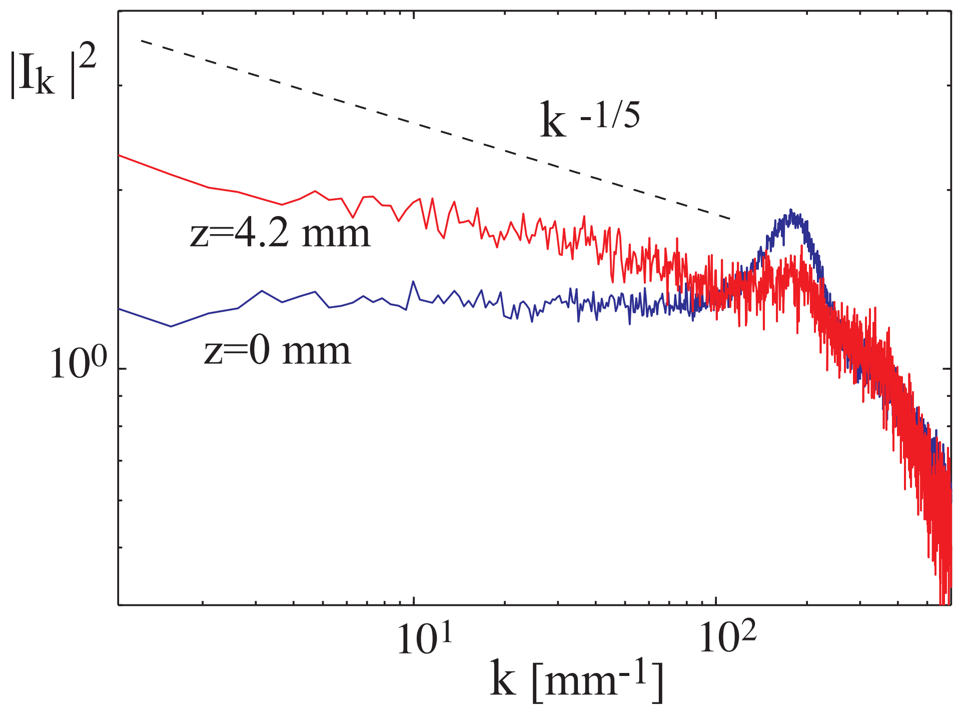


- Indication of an inverse transfer of wave action to large scales
- No agreement with KZ spectrum
- Scaling of the intensity spectrum from KZ:

$$|I_{\mathbf{k}}|^2 = k^{-2x+1} = k^{-1/5}$$

Intensity spectrum

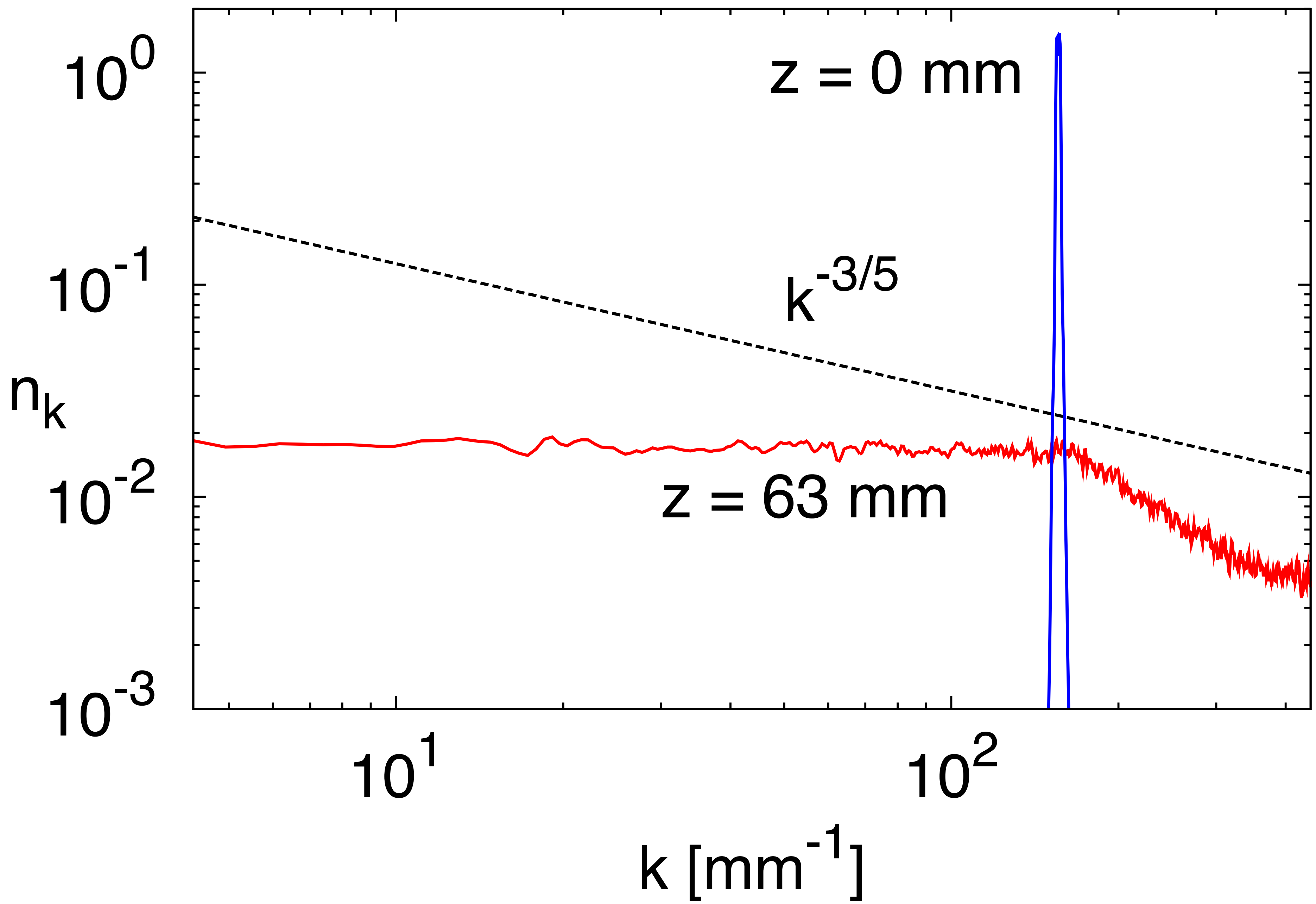




Observation of 'Mixed' Solution?

"Warm cascade" flux on background of the RJ spectrum

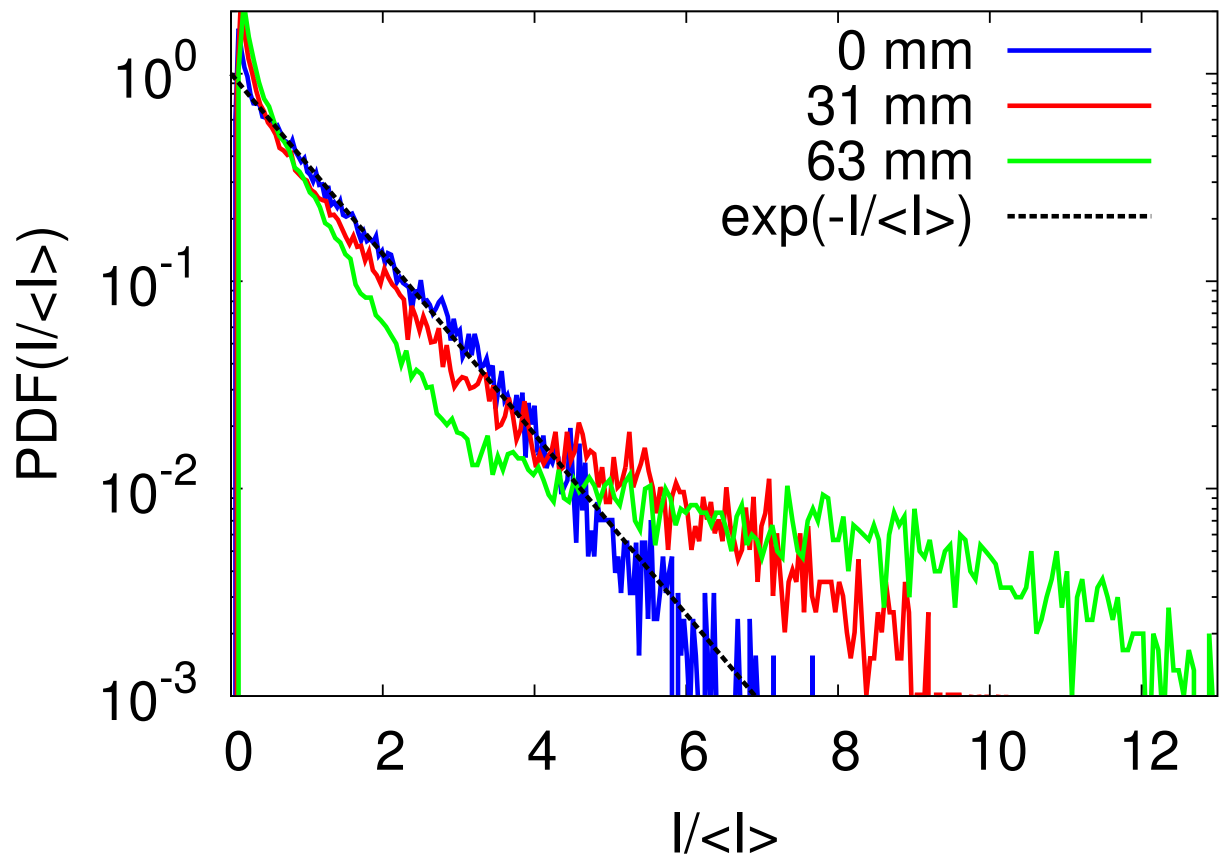
$$n_{\mathbf{k}} = \frac{T}{\omega_k + \mu}$$ n_k tends to const at low k



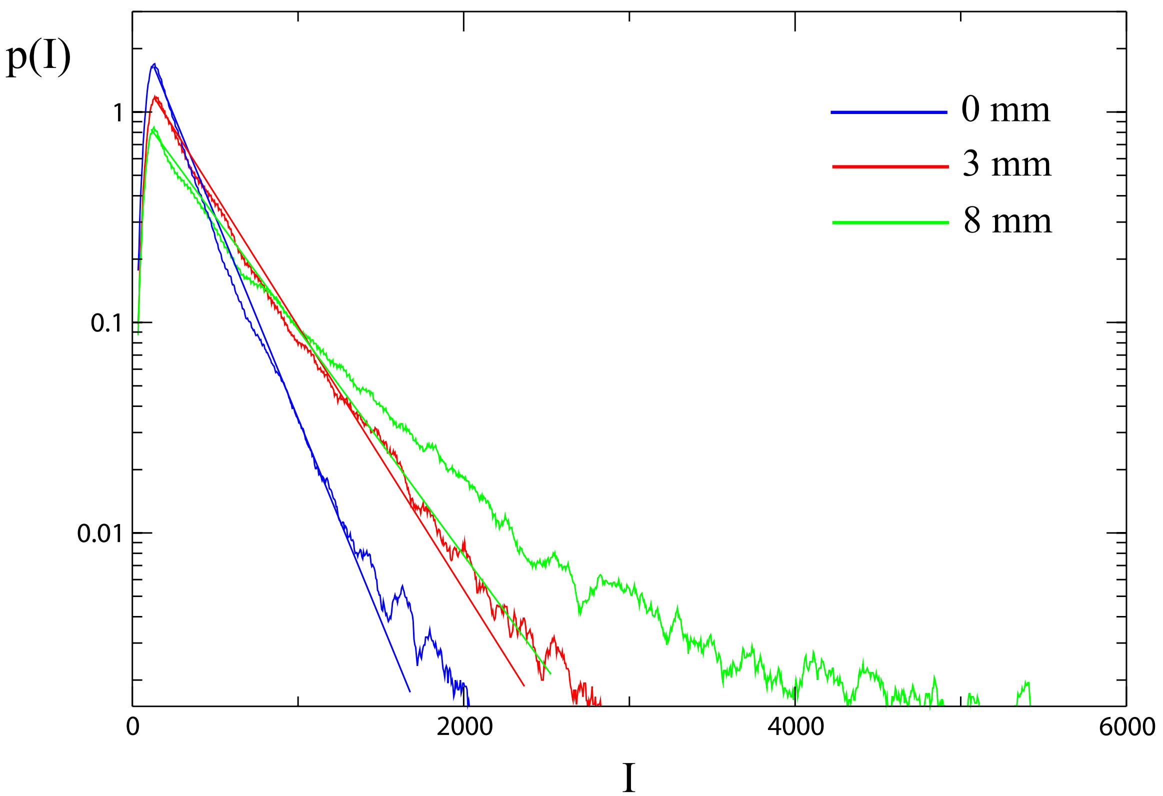
Intensity PDF

Evolution of intensity distribution

Numerical simulation



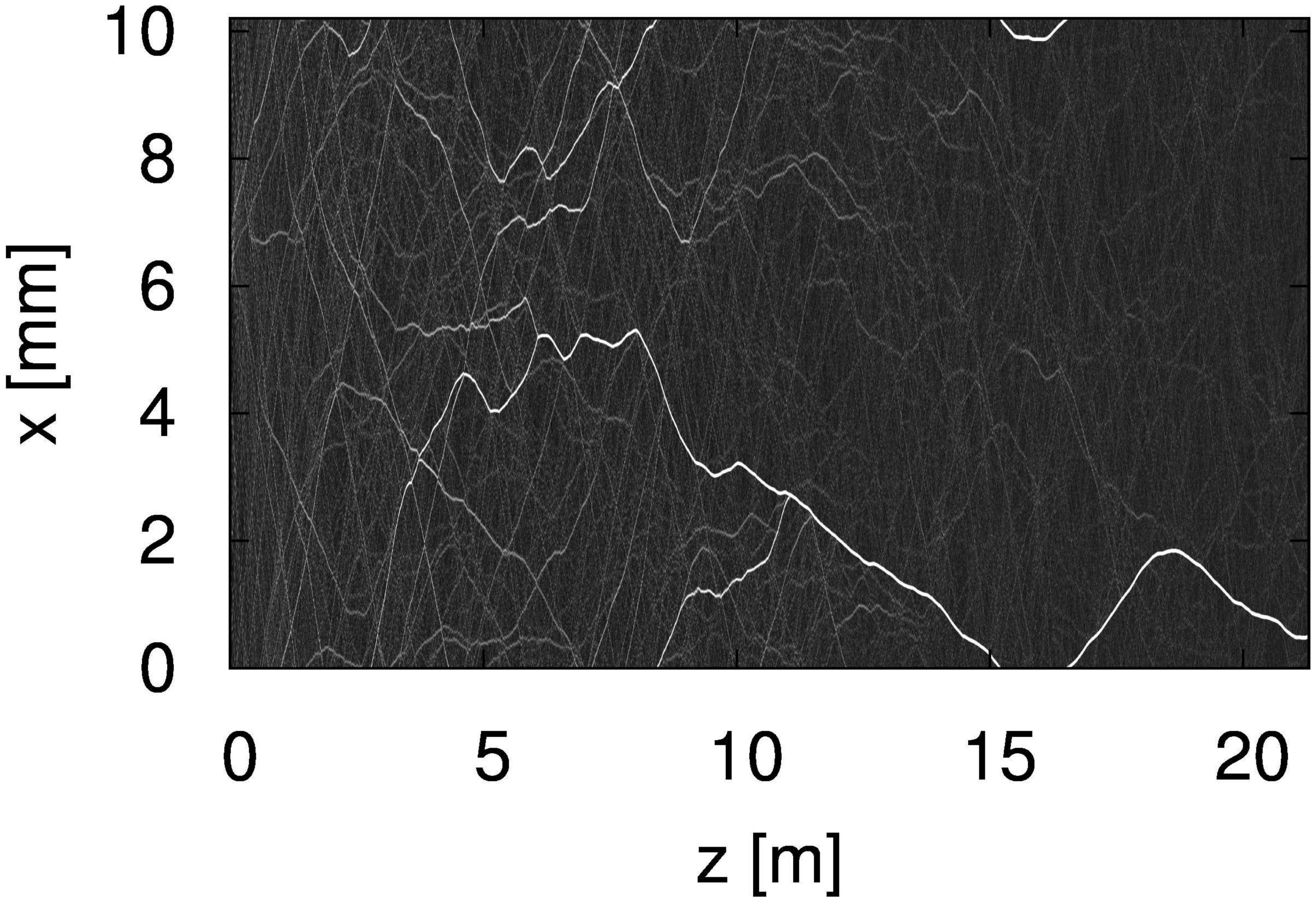
Experiment



Wave turbulence intermittency

- Initial Gaussian statistics evolves to show enhancement of high intensities

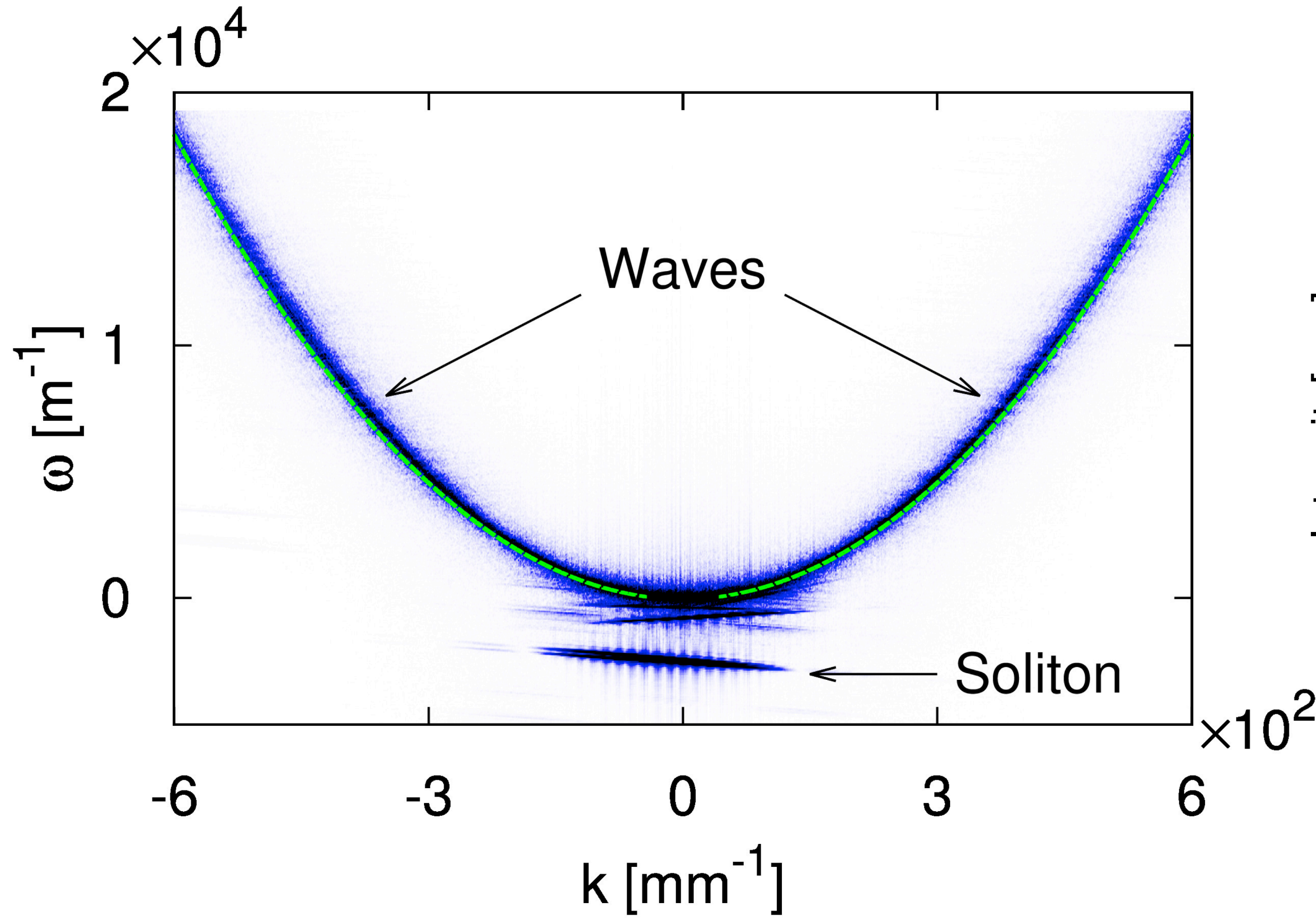
Soliton-Wave Interactions

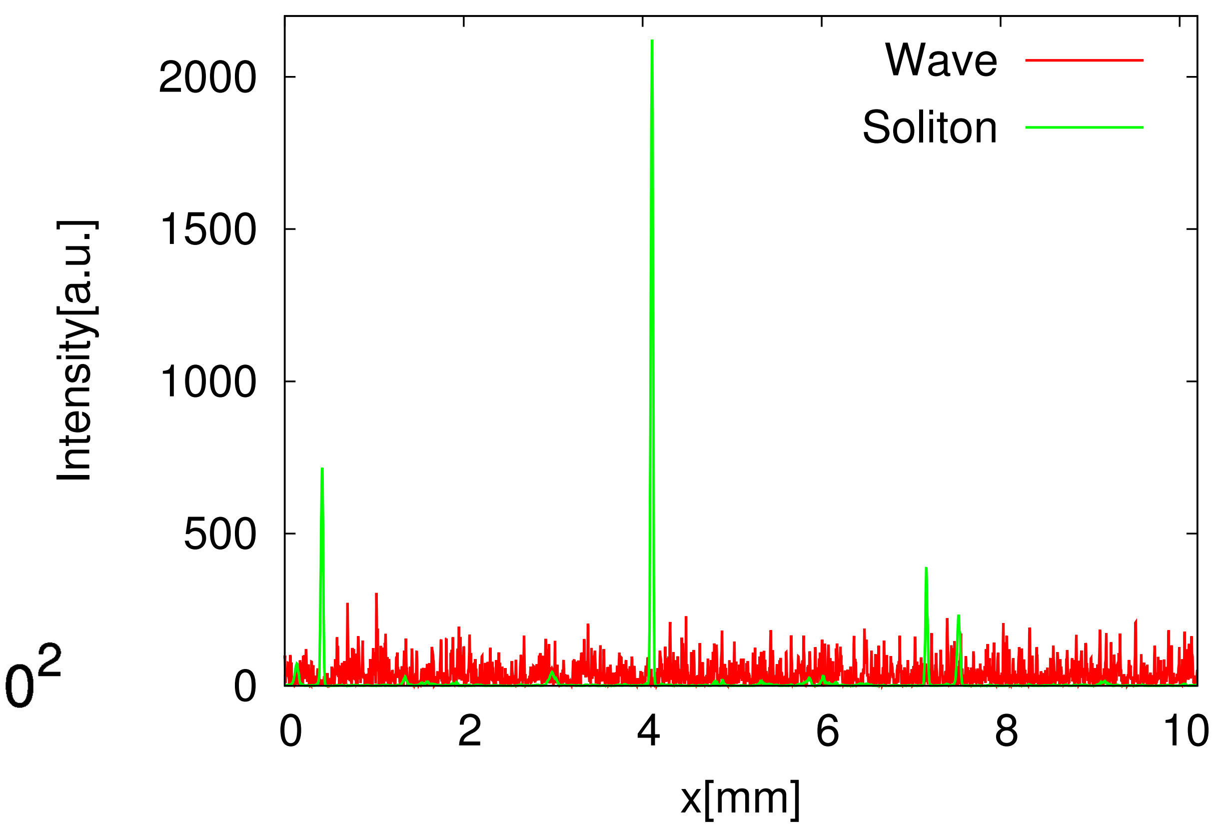


Bogoliubov Dispersion Relation

Presence of condensate modifies linear wave dispersion relation

$$\omega_k = \omega_c + \sqrt{\left(1 + \frac{\varepsilon_0 n_a^4 l_\xi^4 k_0^2 I_0}{2K}\right) k^4 - \frac{\varepsilon_0 n_a^4 l_\xi^2 k_0^2 I_0}{2K} k^2}$$





- Solitons 'peel' from wave background into disjoint coherent structures whose slopes correspond to their velocity

Conclusions

Evidence of an inverse cascade in an optical WT experiment

- Observation of large scale structures from small-scale initial condition

Explanation through wave turbulence theory

- No observation of KZ spectra, but expected due to wrong sign of flux
- Observed “warm cascade” spectrum
- Intensity spectrum scaling in agreement between experiment and numerics

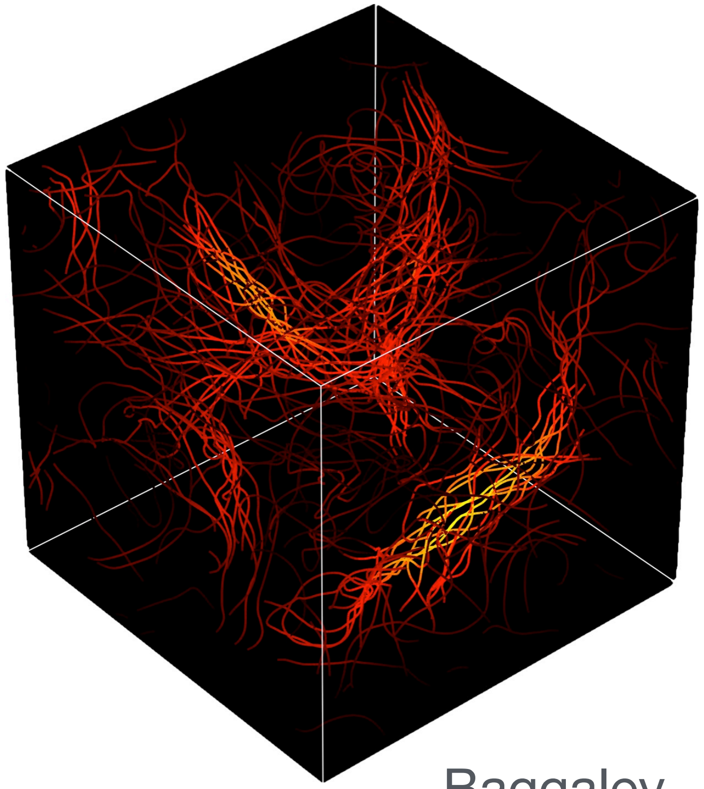
Breakdown of WT regimes through soliton formation via MI

- Appearance of wave turbulence intermittency (long tails)
- Complex soliton-wave dynamics

1. Bortolozzo *et* al. J. Opt. Soc. Am. B, **26**, 2280, (2009)
2. Laurie *et al.* Phys. Reps. **514**, 121, (2012)

BEC turbulence

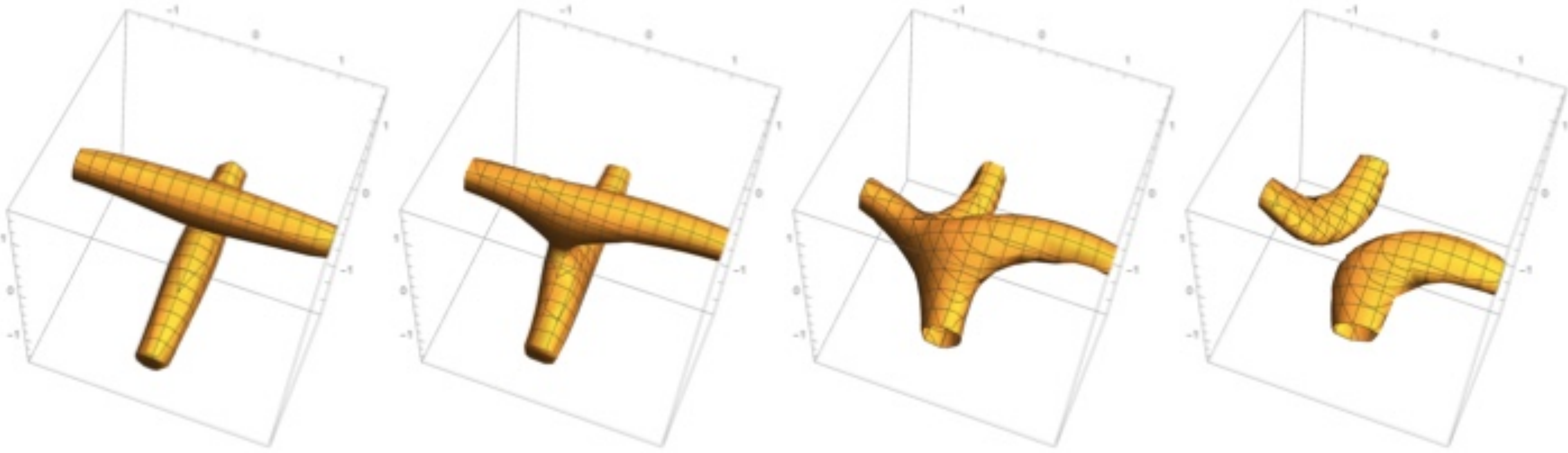
- Before the perfect condensate with a fully coherent phase is reached, the system is abundant with phase defects: so called vortices. At the large scales, these vortices form bundles that **mimic classical eddies.**

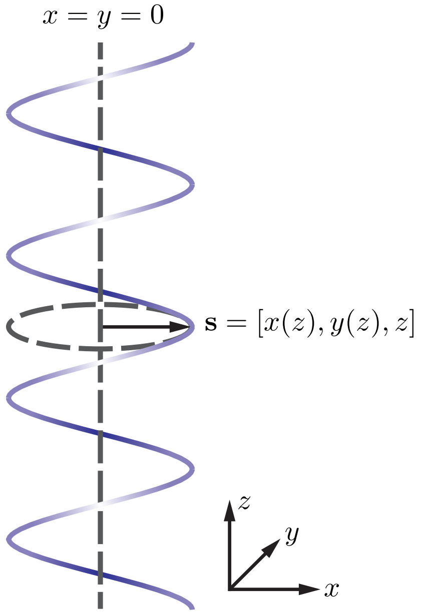


Baggaley (2012)

- Hence a **quasi-classical Kolmogorov** analogy can made with Richardson cascade of quantum vortex bundles

But there is no viscosity, and the cascading energy inevitably reaches the inter-vortex scale. Reconnections occur which give an insignificant amount of turbulent energy to sound. The rest is carried downscale by weakly interacting Kelvin waves





Wave turbulence setup

The Biot-Savart description

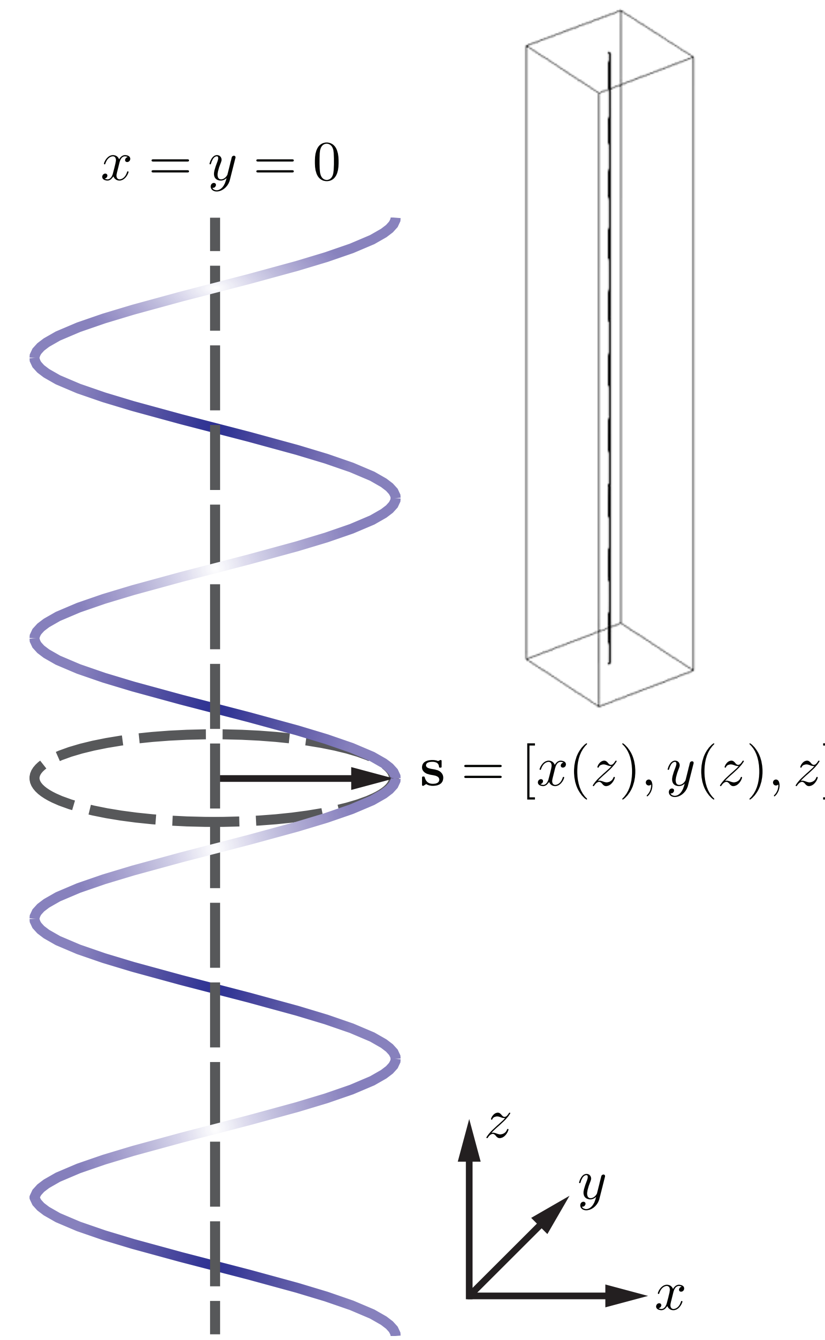
$$\dot{\mathbf{s}} = \frac{\kappa}{4\pi} \oint_{\mathcal{L}} \frac{\mathbf{r}-\mathbf{s}}{|\mathbf{r}-\mathbf{s}|^3} \times \mathrm{d}\mathbf{r}$$

- Consider deviations $\mathbf{s} = [x(z,t), y(z,t), z(t)]$ around **straight vortex line** configuration **periodic** in z

$$a(z,t) = x(z,t) + iy(z,t) \qquad i\kappa \frac{\partial a}{\partial t} = \frac{\delta \mathcal{H}}{\delta a^*}$$

$$\mathcal{H} = \frac{\kappa^2}{4\pi} \int \frac{1 + \mathrm{Re}\left[a'^*(z_1)a'(z_2)\right]}{\sqrt{(z_1 - z_2)^2 + |a(z_1) - a(z_2)|^2}} \mathrm{d}z_1 \mathrm{d}z_2$$

Svistunov, Phys. Rev. B, **52**, 3647, (1995)



Truncation and weak nonlinear expansion

- Regularization of integral by introducing cut-off $\xi < |z_2 - z_1|$
- Expand Hamiltonian in powers of the canonical variable:

$$\epsilon = \frac{|a(z_1) - a(z_2)|}{|z_1 - z_2|} \ll 1 \qquad \mathcal{H} = \mathcal{H}_2 + \mathcal{H}_4 + \mathcal{H}_6 + \cdots$$

Hamiltonian-Fourier representation

Wave action representation of the Hamiltonian

- Introduce wave action variables $a(z,t) = \kappa^{-1/2} \sum_{\mathbf{k}} a_{\mathbf{k}}(t) \exp(i\,\mathbf{k}\,z)$

$$\mathcal{H} = \sum_{\mathbf{k}} \omega_{\mathbf{k}} a_{\mathbf{k}} a_{\mathbf{k}}^* + \frac{1}{4} \sum_{1,2,3,4} T_{3,4}^{1,2}\; a_1 a_2 a_3^* a_4^*\, \delta_{3,4}^{1,2} + \frac{1}{36} \sum_{1,2,3,4,5,6} W_{4,5,6}^{1,2,3}\; a_1 a_2 a_3 a_4^* a_5^* a_6^*\, \delta_{4,5,6}^{1,2,3}$$

$$a_1 = a_{\mathbf{k}_1}(t) \qquad T_{3,4}^{1,2} = T(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4) \qquad \delta_{3,4}^{1,2} = \delta\left(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4\right)$$

Interaction coefficients

$$\omega_{\mathbf{k}} = \frac{\kappa\Lambda}{4\pi}\mathbf{k}^2 - \frac{\kappa}{4\pi}\mathbf{k}^2 \ln(\mathbf{k}\ell_{\text{eff}}), \qquad \Lambda = \ln\left(\ell_{\text{eff}}/\tilde{\xi}\right) \gg 1, \quad \tilde{\xi} = \xi e^{\gamma + \frac{3}{2}}$$

$$T_{3,4}^{1,2} = -\frac{\Lambda}{4\pi}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\mathbf{k}_4 - \frac{1}{16\pi}\left[5\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\mathbf{k}_4 + \mathcal{F}_{3,4}^{1,2}\right]$$

$$W_{4,5,6}^{1,2,3} = \frac{9\Lambda}{8\pi\kappa}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\mathbf{k}_4\mathbf{k}_5\mathbf{k}_6 + \frac{9}{32\pi\kappa}\left[7\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\mathbf{k}_4\mathbf{k}_5\mathbf{k}_6 + \mathcal{G}_{4,5,6}^{1,2,3}\right]$$

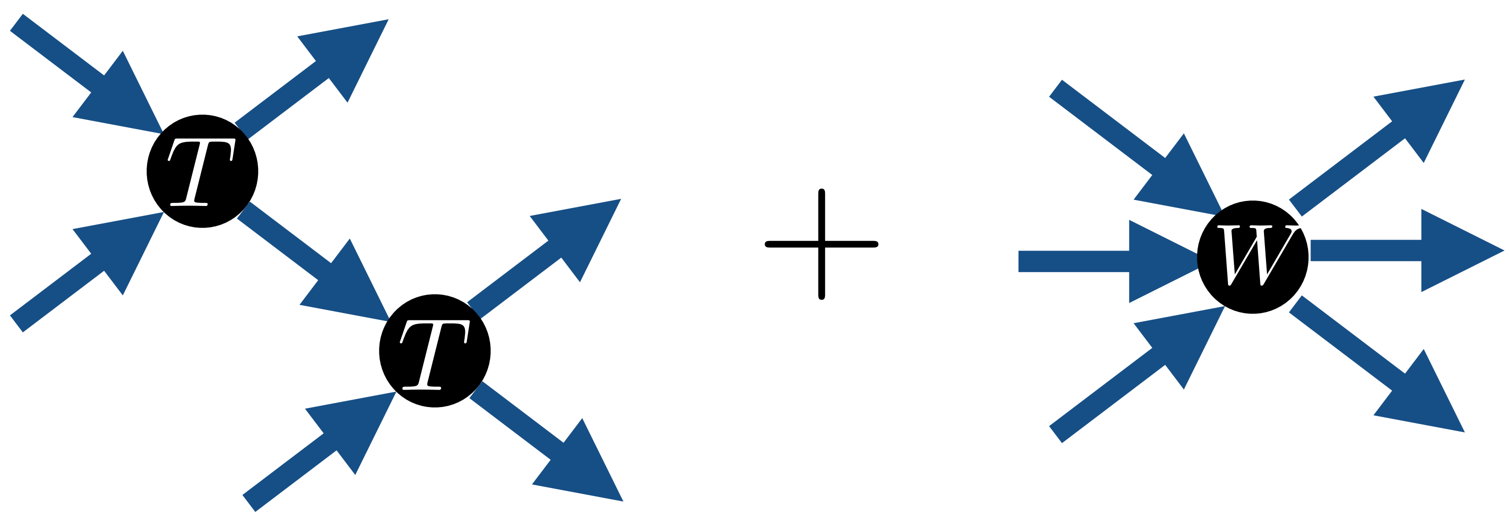
- Separate logarithm divergent terms by introducing an effective length scale ℓ_{eff}
- $\mathcal{F}_{3,4}^{1,2}$ and $\mathcal{G}_{4,5,6}^{1,2,3}$ are terms containing logarithmic contributions

JL *et al.* Phys. Rev. B, **81**, 104526, (2010)

Six-wave interactions

Canonical transformation

- Trivial 4-wave resonances lead to a **nonlinear frequency shift of the linear dynamics**
- A classical canonical transformation $a_{\mathbf{k}} \to c_{\mathbf{k}}$ can be used to express Hamiltonian in new variables so **non-resonant 4-wave terms do no appear**
- Through the transformation, **quartic interactions re-appear as sextic $\mathcal{H}_6$ contributions**



Six-wave interaction coefficient of $\mathcal{H}_6$

JL *et al.* Phys. Rev. B, **81**, 104526, (2010)

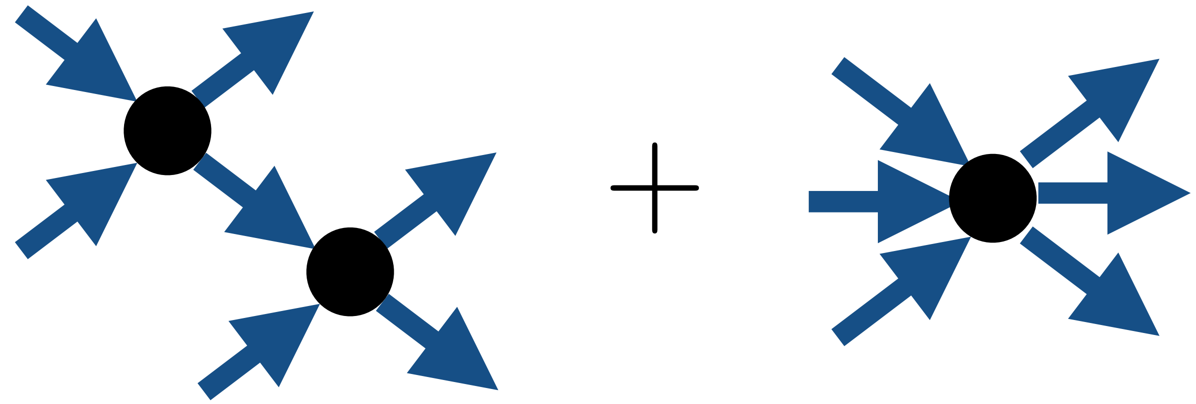
$$\tilde{W} = \underbrace{W^{\Lambda} + \frac{T^{\Lambda} \circ T^{\Lambda}}{\omega^{\Lambda}}}_{=0} \underbrace{+ W^{1} + \frac{T^{1} \circ T^{\Lambda}}{\omega^{\Lambda}} + \frac{T^{\Lambda} \circ T^{1}}{\omega^{\Lambda}} + \frac{T^{\Lambda} \circ T^{\Lambda}}{\left(\omega^{\Lambda}\right)^{2}} \omega^{1}}_{\neq 0} + \mathcal{O}\left(\Lambda^{-1}\right)$$

Divergent terms that correspond to LIA cancel

Leading order terms describing Kelvin-wave dynamics

Kinetic wave equation

Interacting Kelvin-wave dynamics are described by 6-wave processes



The six-wave kinetic equation

- Of particular interest is the **second order correlator** $\langle a_{\mathbf{k}} a^*_{\mathbf{k}_1} \rangle = n_{\mathbf{k}} \delta(\mathbf{k} - \mathbf{k}_1)$
- **Wave energy density** is related to the **wave action** by $E_{\mathbf{k}} = \omega_{\mathbf{k}} n_{\mathbf{k}}$

$$\frac{\partial n_{\mathbf{k}}}{\partial t} = \frac{\epsilon^8 \pi}{6} \int \left| \tilde{W}^{1,2,3}_{4,5,\mathbf{k}} \right|^2 \delta^{1,2,3}_{4,5,\mathbf{k}} \, \delta\left(\omega^{1,2,3}_{4,5,\mathbf{k}}\right) n_1 n_2 n_3 n_4 n_5 n_{\mathbf{k}}$$
$$\times \left[\frac{1}{n_{\mathbf{k}}} + \frac{1}{n_5} + \frac{1}{n_6} - \frac{1}{n_1} - \frac{1}{n_2} - \frac{1}{n_3} \right] \mathrm{d}\mathbf{k}_1 \, \mathrm{d}\mathbf{k}_2 \, \mathrm{d}\mathbf{k}_3 \, \mathrm{d}\mathbf{k}_4 \, \mathrm{d}\mathbf{k}_5$$

Kozik, Svistunov, Phys. Rev. Lett. **92**, 035301, (2004)
JL *et al.* Phys. Rev. B, **81**, 104526, (2010)

Steady-state Kolmogorov-Zakharov solutions $\frac{\partial n_{\mathbf{k}}}{\partial t} = 0$

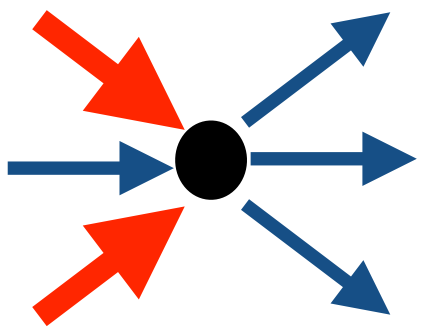
- **Zakharov transformation** of KE to determine power-law steady state solutions
- One solution corresponds to **constant energy transfer to small scales** by Kelvin-waves

$$\boxed{E_k = C_{KS} \, \Lambda \, \kappa^{7/5} \, \epsilon^{1/5} \, k^{-7/5}}$$

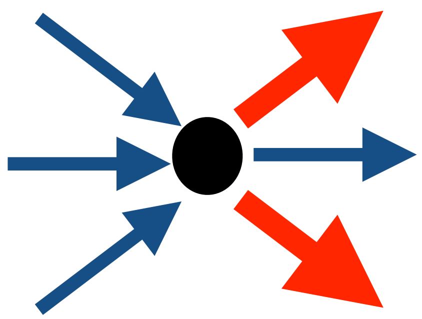
Kozik-Svistunov energy spectrum

Effective four-wave description

Nonlocal six-wave interactions



- **Exact calculation** of **Kinetic equation** shows that it **diverges** in the limit of two long Kelvin-waves
- Effective four-wave interaction takes place on **curved** vortex line **derived** from **nonlocal six-wave process**



JL *et al.* Phys. Rev. B, **81**, 104526, (2010)
L'vov, Nazarenko, Low Temp. Phys. **36**, 785, (2010)

Effective four-wave kinetic equation

$$\frac{\partial n_{\mathbf{k}}}{\partial t} = \frac{\epsilon^8 \pi}{12} \int \Bigg\{ |V_{\mathbf{k}}^{1,2,3}|^2 n_1 n_2 n_3 n_{\mathbf{k}} \left[\frac{1}{n_{\mathbf{k}}} - \frac{1}{n_1} - \frac{1}{n_2} - \frac{1}{n_3} \right] \delta^{\mathbf{k}}_{1,2,3} \delta\left(\omega^{\mathbf{k}}_{1,2,3}\right)$$
$$+ 3 |V_1^{\mathbf{k},2,3}|^2 n_1 n_2 n_3 n_{\mathbf{k}} \left[\frac{1}{n_1} - \frac{1}{n_{\mathbf{k}}} - \frac{1}{n_2} - \frac{1}{n_3} \right] \delta^{1}_{\mathbf{k},2,3} \delta\left(\omega^{1}_{\mathbf{k},2,3}\right) \Bigg\} \, \mathrm{d}\mathbf{k}_1 \, \mathrm{d}\mathbf{k}_2 \, \mathrm{d}\mathbf{k}_3$$

New Kolmogorov-Zakharov solution

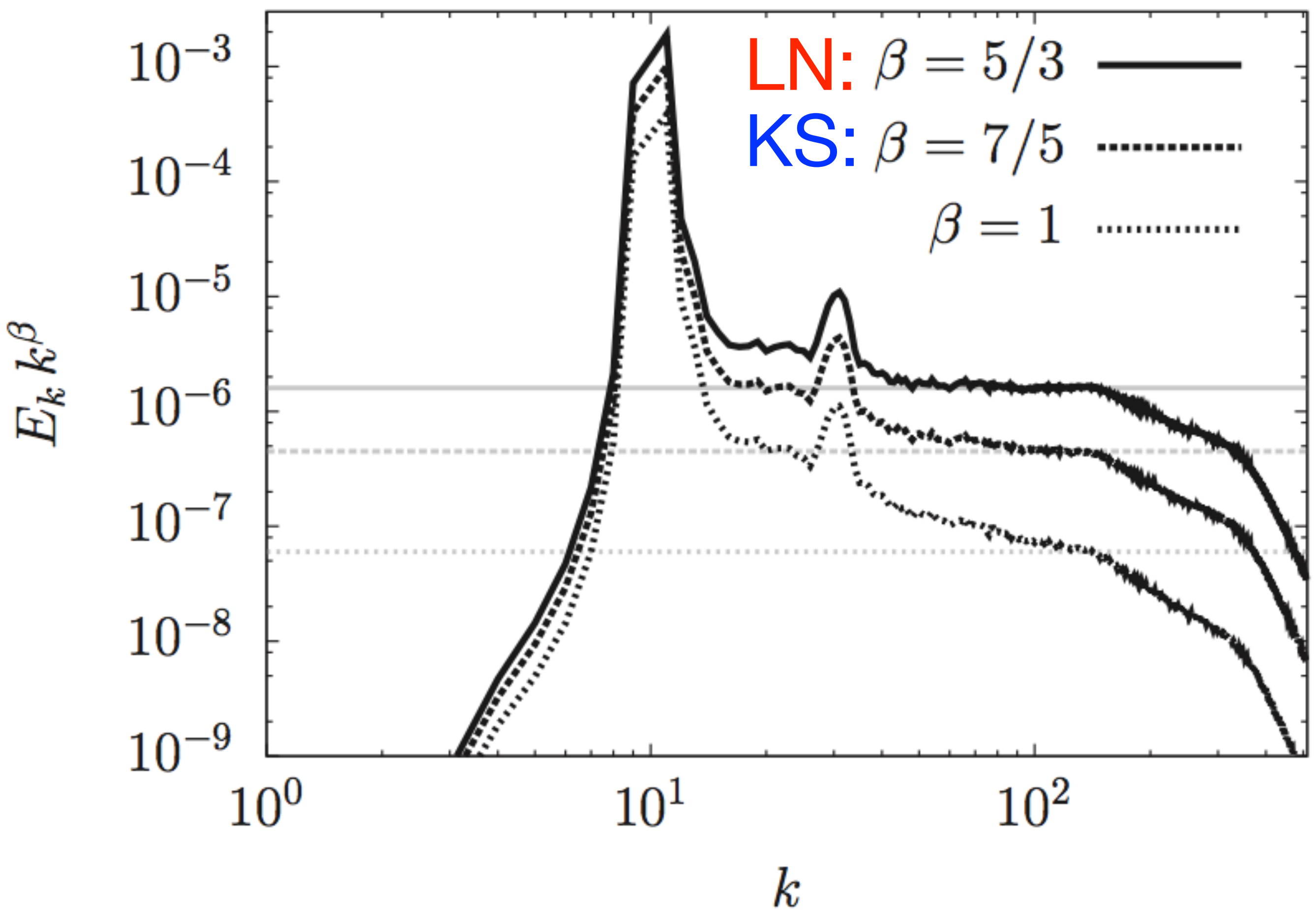
$$\boxed{E_k = C_{LN} \, \Lambda \, \kappa^{7/5} \, \epsilon^{1/3} \, \Psi^{-2/3} \, k^{-5/3}}$$

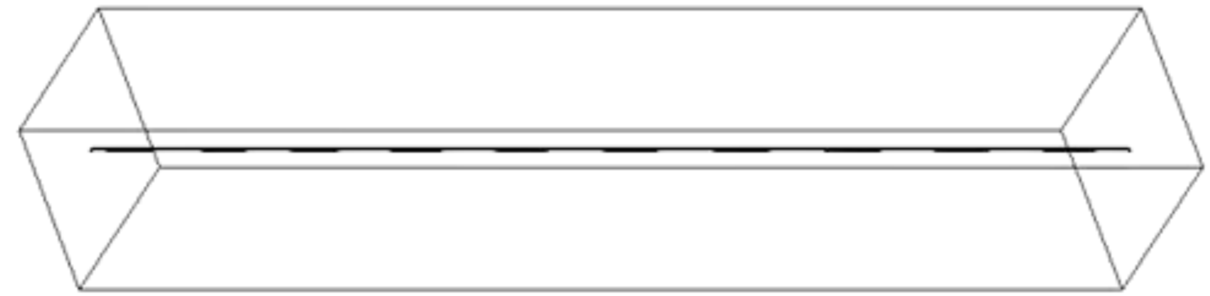
L'vov-Nazarenko energy spectrum

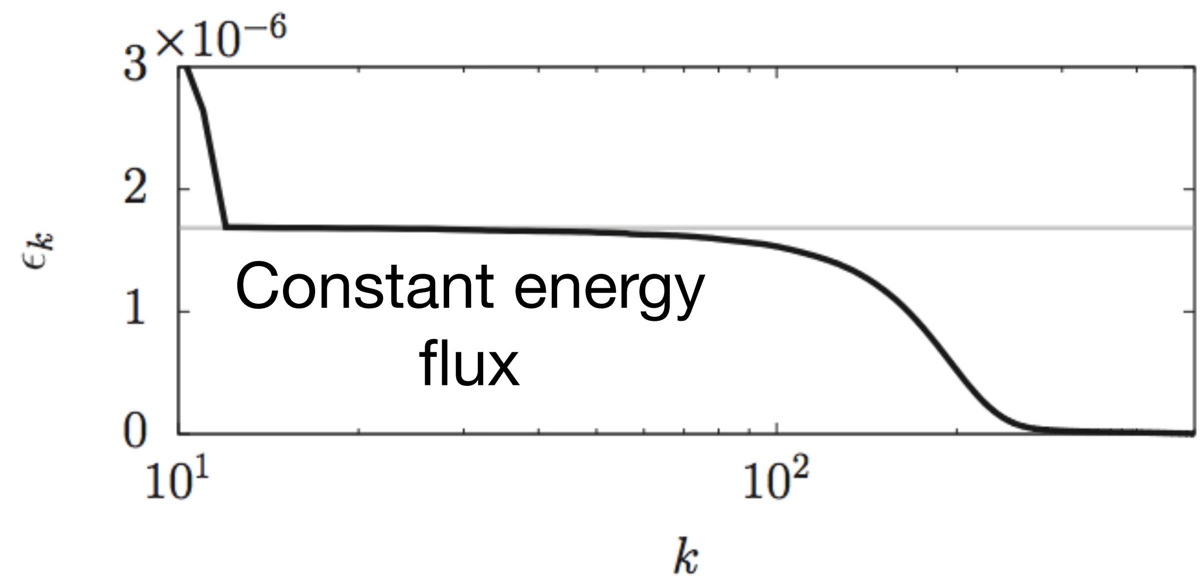
Identification of spectrum: Biot-Savart

Biot-Savart model for superfluid helium-4

$$\dot{\mathbf{s}} = \frac{\kappa}{4\pi} \oint_{\mathcal{L}} \frac{\mathbf{r} - \mathbf{s}}{|\mathbf{r} - \mathbf{s}|^3} \times d\mathbf{r} + \mathbf{F} - \mathbf{D}$$







Baggaley, JL, Phys. Rev. B, **94**, 025301, (2014)

Numerically measured spectrum prefactor

4-Wave LN-theory: $C_{LN}^{num} = 0.318$

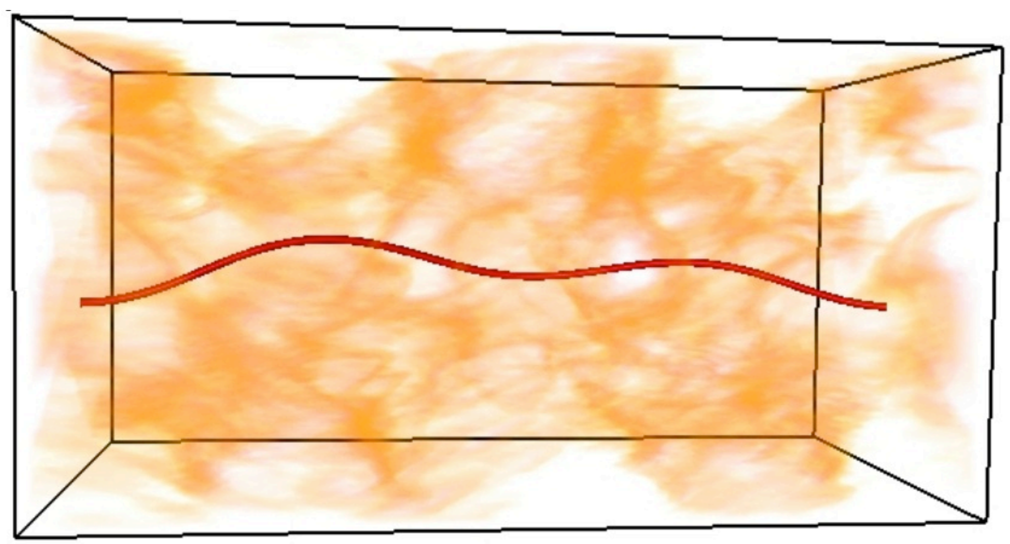
- 4-wave LN prefactor within **5%** of **theoretical prediction** $C_{LN} = 0.304$

Identification of spectrum: GP Eqn.

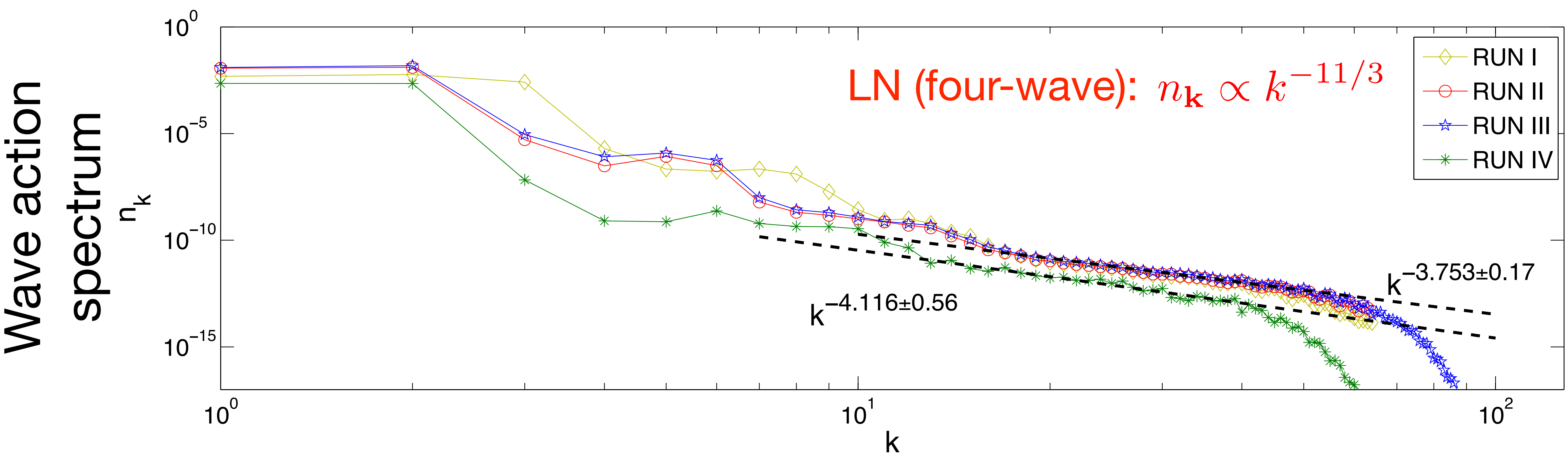
Gross-Pitaevskii model for BECs

$$\boxed{i\dot{\Psi} = -\nabla^2\Psi + \Psi\left|\Psi\right|^2}$$

1. Decaying vortex line simulation



Krstulović, Phys. Rev. E, **86**, 055301, (2012)

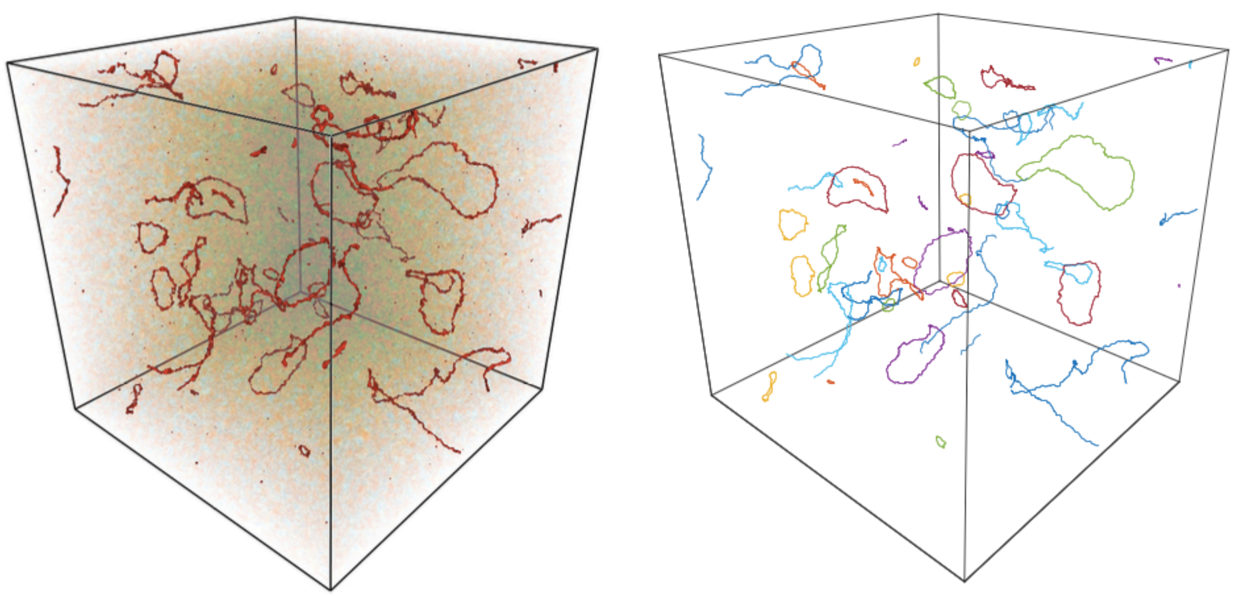


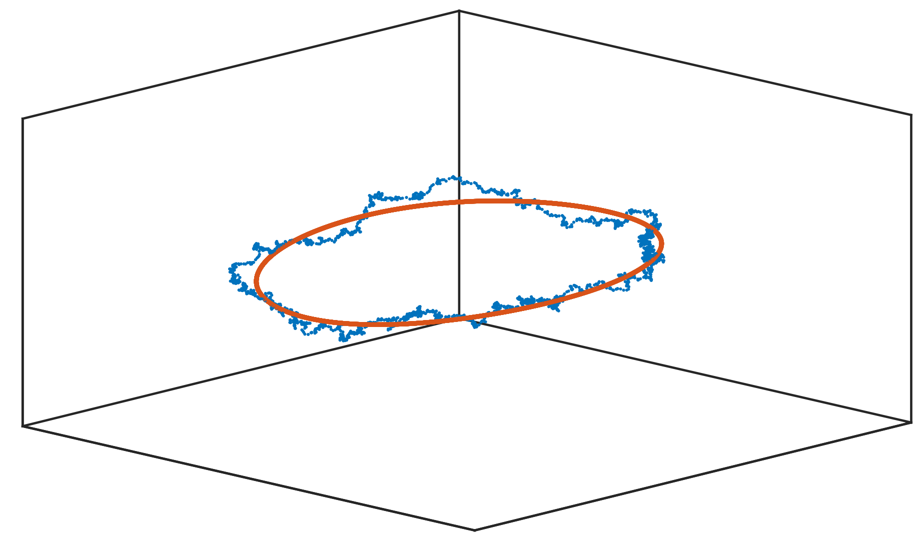
Identification of spectrum: GP Eqn.

Gross-Pitaevskii model for BECs

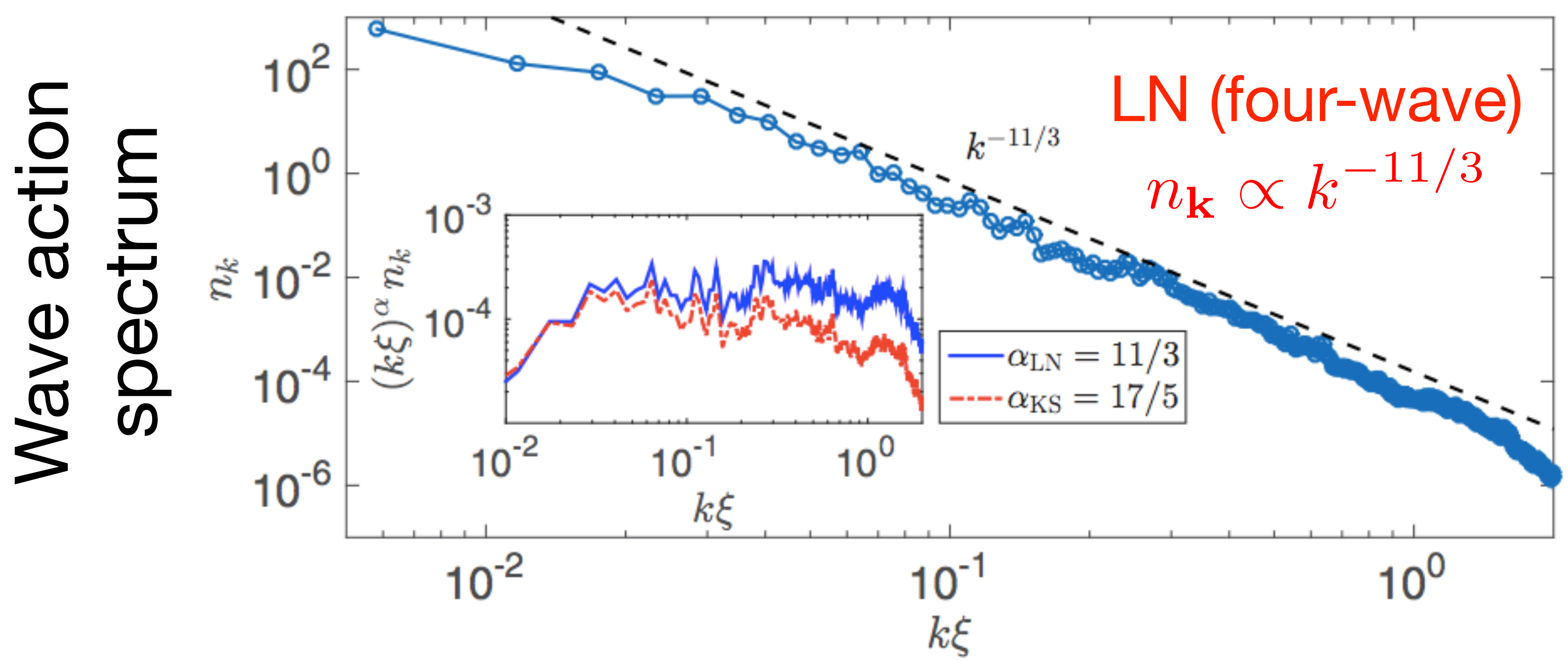
$$\boxed{i\dot{\Psi} = -\nabla^2\Psi + \Psi\,|\Psi|^2}$$

2. Vortex tangle simulation





Villois *et al.* Phys. Rev. E, **93**, 061103(R), (2016)



Conclusions and perspectives

Conclusions

- For QT, **weakly nonlinear Kelvin-wave interactions** are consider as the **primary energy transfer mechanism** at small-scale in zero temperature limit
- **Wave Turbulence theory** can systematically describe Kelvin-waves on a single periodic vortex line described by the **Biot-Savart model**
- **Effective four-wave description** derived from **nonlocal** six-wave interaction
- Energy spectrum **scaling and prefactor** can be **exactly obtained from theory**
- Numerical simulations in both **Biot-Savart** and **Gross-Pitaevskii** equations **validate the WT prediction**