Evolution of non-Gaussian wave fields

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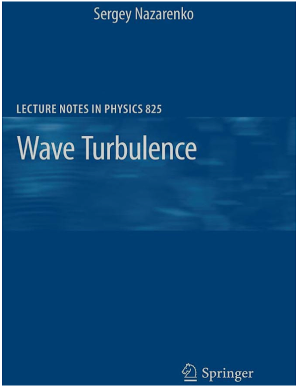
Outline

Most of WT theory considers the wave spectrum. In my book I summarise an approach considering evolution of the probability density functions (PDFs) of the wave amplitudes and derive evolution equation for these objects. Here, I will discuss a general non-stationary solution for the 1-mode PDF and derive conditions for achieving Gaussianity.



Master example: Gross-Pitaevskii equation

Gross-Pitaevskii equation describes waves in BEC and in nonlinear optics:

$$i\,\frac{\partial \psi}{\partial t} + \nabla^2 \psi - |\psi|^2 \psi = 0\ . \tag{1}$$

where $\psi = \psi(\mathbf{x}, t)$ is a complex function of two or three space coordinates and time t.

Fourier space

Let us consider a periodic system, $\mathbf{x} \in \mathbb{T}^d$, with period L in all directions ($d = 2$ or 3). Using Fourier coefficients

$$a_{\mathbf{k}}(t) = \frac{1}{L^d} \int_{\mathrm{Box}} \psi(\mathbf{x}.t) e^{-i\mathbf{k}\cdot\mathbf{x}} d\mathbf{x} \ , \tag{2}$$

rewrite the GP equation as

$$i\dot{a}_k = \omega_k a_k + i \sum_{1,2,3} a_1 a_2 a_3 \delta^{3k}_{12}, \tag{3}$$

where the frequency of this wave is

$$\omega_k = k^2 \ . \tag{4}$$

Here, $a_{1,2,3} \equiv a_{\mathbf{k}_{1,2,3}}$, and $\delta^{3k}_{12} = \delta(\mathbf{k}_3 + \mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2)$.
In the linear limit,

$$a_k = A_k e^{-i\omega_k t} \ , \tag{5}$$

where $A_k \in \mathbb{C}$ is a time-independent amplitude of the wave.

Discrete $\mathbf{k}$-space

Since we consider a periodic system, $\mathbf{x} \in \mathbb{T}^d$, the wavenumbers are discrete, $\mathbf{k} \in \mathbb{Z}^d$. Let the total number of modes be finite and bounded by some $k_{\max}$ (eg. a dissipation cutoff at high wavenumbers). Denote by B_N the set of all wavenumbers $\mathbf{k}$ inside the $\mathbf{k}$-space box of volume $(2k_{\max})^d$:

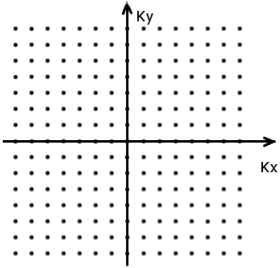


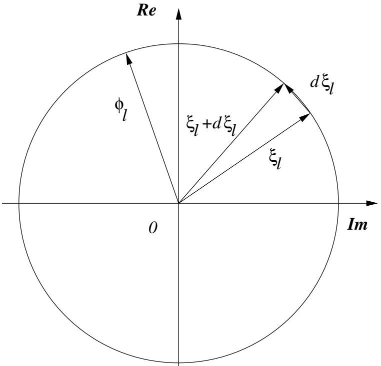
Figure: Set of active wave modes, $B_N \subset \mathbb{Z}^d$.

Amplitude-Phase representation

We write the wave function in terms of its amplitude and phase

$$a(\mathbf{k}, t) = \sqrt{J_{\mathbf{k}}}\phi_{\mathbf{k}} \ ,$$

where $J_{\mathbf{k}} \in \mathbb{R}^{+}$ is the intensity and $\phi_{\mathbf{k}} \in \mathbb{S}^{1}$ is the phase factor of the mode $\mathbf{k}$. By $\mathbb{S}^{1}$ we mean the unit circle in the complex plane, i.e. $\phi_{\mathbf{k}} = e^{i\varphi_{\mathbf{k}}}$:



Probability Density Functions.

Denote the set of all $J_{\mathbf{k}}$ and $\phi_{\mathbf{k}}$ with $\mathbf{k}$ such that $\mathbf{k} \in B_N$ as $\{J, \phi\}$.
The probability of finding $J_{\mathbf{k}}$ inside $(s_{\mathbf{k}}, s_{\mathbf{k}} + ds_{\mathbf{k}}) \subset \mathbb{R}^+$ and finding $\phi_{\mathbf{k}}$ on the arch $(\xi_{\mathbf{k}}, \xi_{\mathbf{k}} + d\xi_{\mathbf{k}}) \subset \mathbb{S}^1$ (see Figure) is given in terms of the joint PDF $\mathcal{P}^{(N)}\{s, \xi\}$ as

$$\mathcal{P}^{(N)}\{s, \xi\} \prod_{\mathbf{k} \in B_N} ds_{\mathbf{k}} |d\xi_{\mathbf{k}}| \ . \tag{6}$$

Single-mode amplitude PDF, $\mathcal{P}^{(1,a)} \equiv \mathcal{P}^{(a)}_{\mathbf{k}}(s_{\mathbf{k}})$, is obtained via integrated out all phases ξ, and all amplitudes s but one, $s_{\mathbf{k}}$.

RP fields.

Definition. Random phase (RP) field: all ϕ are independent random variables (i.r.v.) each uniformly distributed on $\mathbb{S}^1$.
Thus for a RP field

$$\mathcal{P}^{(N)}\{s,\xi\} = \frac{1}{(2\pi)^N}\mathcal{P}^{(N,a)}\{s\} \ .$$

Note: RP (in addition to the weak nonlinearity) is enough for the lowest level WT closure leading to an equation for the N-point amplitude-only PDF. However, it is not sufficient for the one-point WT closure, in particular the wave kinetic equation, and we need to assume something about the amplitudes too.

Random Phase and Amplitude (RPA) field definition

1. All the amplitudes and all the phases are i.r.v.,
2. All the phases are uniformly distributed on $\mathbb{S}^1$,
3. For RPA fields, the PDF has a product-factorized form,

$$\mathcal{P}^{(N)}\{s,\xi\} = \frac{1}{(2\pi)^N} \prod_{\mathbf{k}_l \in B_N} \mathcal{P}_j^{(a)}(s_j). \tag{7}$$

We have changed the standard meaning of RPA which usually stands for "Random phase approximation". In our definition of RPA:

1. The amplitudes are random, not only the phases.
2. RPA is defined as a property of the field, not an approximation.

RPA does not mean Gaussianity because it does not specify $\mathcal{P}_j^{(a)}(s_j)$. For Gaussian fields $\mathcal{P}^{(a)}(s_j) = \frac{1}{\langle J_j \rangle} \exp\left[-\frac{s_j}{\langle J_j \rangle}\right]$. WT does not require Gaussianity, only RPA, so we can study non-Gaussian fields!

Wave spectrum

The *wave spectrum* is defined as follows

$$n_k = (L/2\pi)^d \langle J_k \rangle \,.$$

For the infinite-box limit,

$$\langle \psi_k, \psi^*_{k'} \rangle = n_k \delta(\mathbf{k} - \mathbf{k}') \ ,$$

where $\delta(\mathbf{x})$ is the Dirac's delta function.
In terms of the generating function and the PDF, the wave spectrum can be expressed as follows,

$$n_k = \int_0^\infty s_k \mathcal{P}^{(a)}(s_k) ds_k \ .$$

Assumptions in the wave turbulence theory

- Weak nonlinearity.
- Initial RP statistics.

Equation for the PDF.

We have the following equation for the PDF,

$$\dot{\mathcal{P}} = 8\pi \int \delta(\omega^{jl}_{mn})\delta^{jl}_{mn} \left[\frac{\delta}{\delta s}\right]_4 \left(s_j s_l s_m s_n \left[\frac{\delta}{\delta s}\right]_4 \mathcal{P}\right) d\mathbf{k}_j d\mathbf{k}_l d\mathbf{k}_m d\mathbf{k}_n,$$

$$\left[\frac{\delta}{\delta s}\right]_4 = \frac{\delta}{\delta s_j} + \frac{\delta}{\delta s_l} - \frac{\delta}{\delta s_m} - \frac{\delta}{\delta s_n}.$$

No phases: phase randomness propagated. Amplitudes not separated: amplitude randomness only in coarse-grained sense.

Multiplying by s_k and integration over all s_j, we get the kinetic equation:

$$\dot{n}_k = 4\pi \int n_{\mathbf{k}_1} n_{\mathbf{k}_2} n_{\mathbf{k}_3} n_{\mathbf{k}} \left[\frac{1}{n_{\mathbf{k}}} + \frac{1}{n_{\mathbf{k}_3}} - \frac{1}{n_{\mathbf{k}_1}} - \frac{1}{n_{\mathbf{k}_2}}\right] \times \delta(\omega_k + \omega_{k_3} - \omega_{k_1} - \omega_{k_2})\delta(\mathbf{k} + \mathbf{k}_3 - \mathbf{k}_1 - \mathbf{k}_2)\, d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3.$$

Solutions for the PDF

An arbitrary function of the un-averaged energy $E = \int \omega_k s_k \, d\mathbf{k}$ is is a steady solution,

$$\dot{\mathcal{P}}(E) = 0.$$

This property is common for all Liouville-type N-particle equations. An important special case is given by the exponential function,

$$\mathcal{P} = e^{-\beta \int \omega_k s_k \, d\mathbf{k}},$$

where β is an arbitrary constant. To understand the meaning of this solution, let us write its discrete version:

$$\mathcal{P} = \prod_j^N e^{-\beta \omega_j s_j}.$$

This solution describes a thermodynamic equilibrium, corresponding to N statistically independent Gaussian-distributed modes with a mean intensity given by a Rayleigh-Jeans spectrum, $n_k \sim 1/\omega_k$.

Evolution of the one-mode PDF

Evolution equation for the one-mode PDF:

$$\frac{\partial P(t,s(k))}{\partial t} + \frac{\partial F}{\partial s(k)} = 0, \tag{8}$$

with the probability flux

$$F = -s\Big(\gamma P + \eta \frac{\partial P}{\partial s}\Big) \tag{9}$$

and

$$\begin{aligned} \eta_k(t) &= 4\pi\epsilon^2 \int \delta(\omega^{jl}_{mn})\delta^{jl}_{mn} n_1 n_2 n_3 dk_1 dk_2 dk_3, \\ \gamma_k(t) &= 8\pi\epsilon^2 \int \delta(\omega^{jl}_{mn})\delta^{jl}_{mn} \Big[n_1(n_2+n_3) - n_2 n_3\Big] dk_1 dk_2 dk_3. \end{aligned}$$

Gaussian fields with $P = \frac{1}{n_k}\exp\left[-\frac{s_k}{n_k}\right]$ satisfy the stationary equation (with $F=0$). Can we find a solution for evolving systems?

Non-stationary solutions

Theorem

1. Wave fields which are Gaussian initially will remain Gaussian for all time.
2. Wave turbulence asymptotically becomes Gaussian if

$$\lim_{t\to\infty} \frac{n(0)e^{-\int_0^t \gamma(t')dt'}}{n(t)} = 0. \tag{10}$$

Remarks:

1. Condition (10) is satisfied for the inertial range modes in forced-dissipated systems which tend to a steady state. Indeed, in this case $\gamma \to \eta/n$ which is a positive constant (at fixed mode $\mathbf{k}$), so the time integral of this quantity diverges as $t \to \infty$.
2. In absence of forcing and dissipation, spectrum $n_{\mathbf{k}}$ decays to zero at any mode $\mathbf{k}$ as $t \to \infty$, and so does $\gamma_{\mathbf{k}}$. Thus the integral of $\gamma_{\mathbf{k}}(t)$ may converge as $t \to \infty$, which means that non-Gaussianity of some (or all) wave modes may persist as $t \to \infty$.
3. In general, function $\gamma_{\mathbf{k}}(t)$ is not sign definite, and there may be transient time periods where $\gamma_{\mathbf{k}}(t) < 0$. The deviation from Gaussianity of some (or all) wave modes may increase during these periods.

General time-dependent solution

We have:

$$\mathcal{P}(t,s) = \int_0^\infty \mathcal{P}(0,J)\mathcal{P}_J(t,s)dJ. \quad (11)$$

where the Green function is

$$\mathcal{P}_J(t,s) = \frac{1}{\tilde{n}} e^{-\frac{s}{\tilde{n}} - a\tilde{n}} I_0(2\sqrt{as}), \quad (12)$$

$$\tilde{n} = n(t) - Je^{-\int_0^t \gamma(t')dt'} \quad (13)$$

(note that $n(0) = J$ and $\tilde{n}(0) = 0$), $a = \frac{J}{\tilde{n}^2} e^{-\int_0^t \gamma(t')dt'}$ and $I_0(x)$ is the zeroth modified Bessel function of the first kind. Since $I_0(0) = 1$, so we recover $\mathcal{P}_\delta \to \mathcal{P}_G = \frac{1}{n} e^{-s/n}$ as $t \to \infty$ if condition (10) is satisfied provided that s is not too large, $as \ll 1$.

Asymptotic behaviour

Now let us suppose that condition (10) is satisfied and let consider the asymptotic behaviour of the probability distribution at large s and large t, and $as \gg 1$ (i.e. s is much larger than $1/a$ which is itself large). Taking into account that $I_0(x) \xrightarrow{x\to\infty} \frac{e^x}{\sqrt{2\pi x}}$, we have:

$$\mathcal{P}_J(s,t) \to \frac{\mathcal{P}_G}{(2\pi)^{1/2}(as)^{1/4}} e^{2\sqrt{as}-as} \ll \mathcal{P}_G \quad \text{for} \quad as \gg 1, \ \int_0^t \gamma(t')dt' \gg 1. \tag{14}$$

Thus, we see a front at $s \sim s^*(t) = 1/a$ moving toward large s as $t \to \infty$. The PDF ahead of this front is depleted with respect to the Gaussian distribution, whereas behind the front it asymptotes to Gaussian. Obviously, the same kind of behaviour will be realised for any solution (11) arising from initial data having a finite support in s.

Conclusions

- For the inertial range modes in forced-dissipated systems approaching a steady state, the Gaussian statistics will form at $t \to \infty$.
- Since, the typical evolution times are the same for $n_{\mathbf{k}}$ and for the PDF, the latter will remain non-Gaussian over a substantial time if the initial field is non-Gaussian. Such situations are typical in natural conditions with non-Gaussian forcing and in numerics with initially deterministic intensities.
- In absence of forcing and dissipation, non-Gaussianity may persist as $t \to \infty$. Furthermore, the deviation from Gaussianity of some (or all) wave modes may increase during transient periods.