Turbulence of weak gravitational waves in the early Universe

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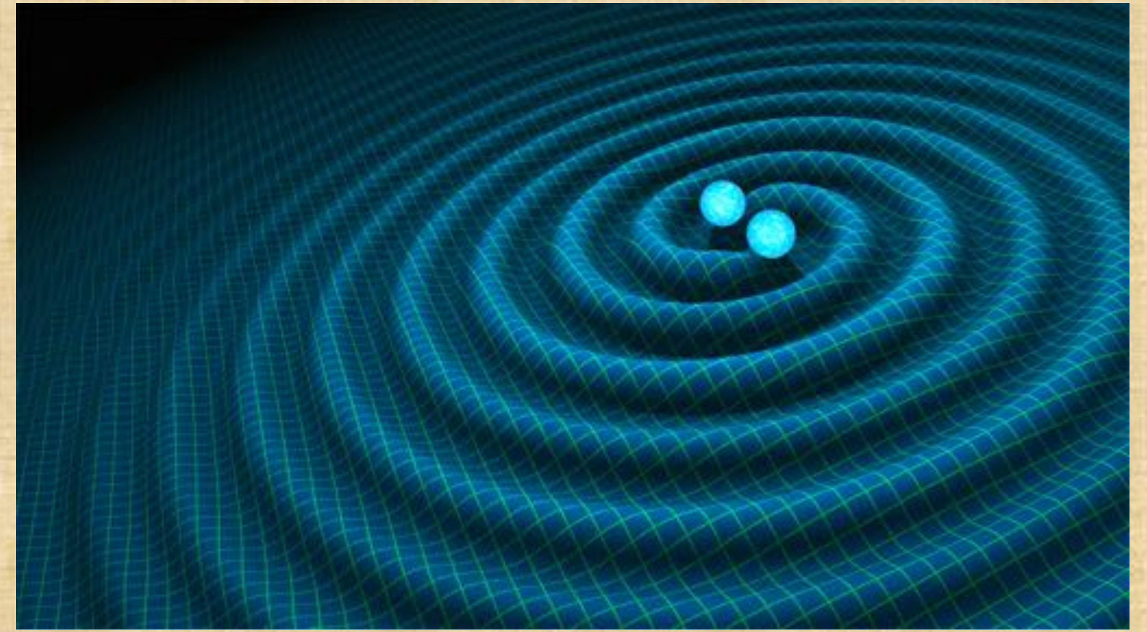
Observation of Gravitational Waves from a Binary Black Hole Merger

B. P. Abbott *et al.**

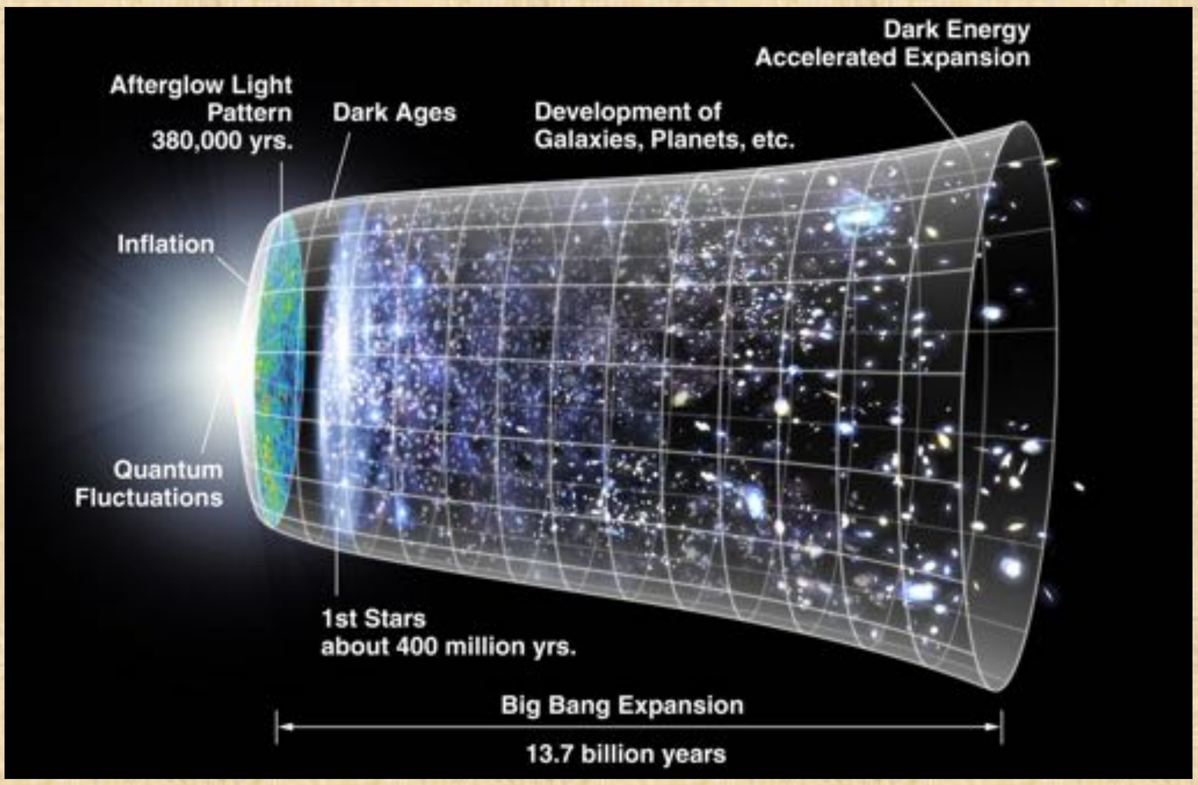
(LIGO Scientific Collaboration and Virgo Collaboration)

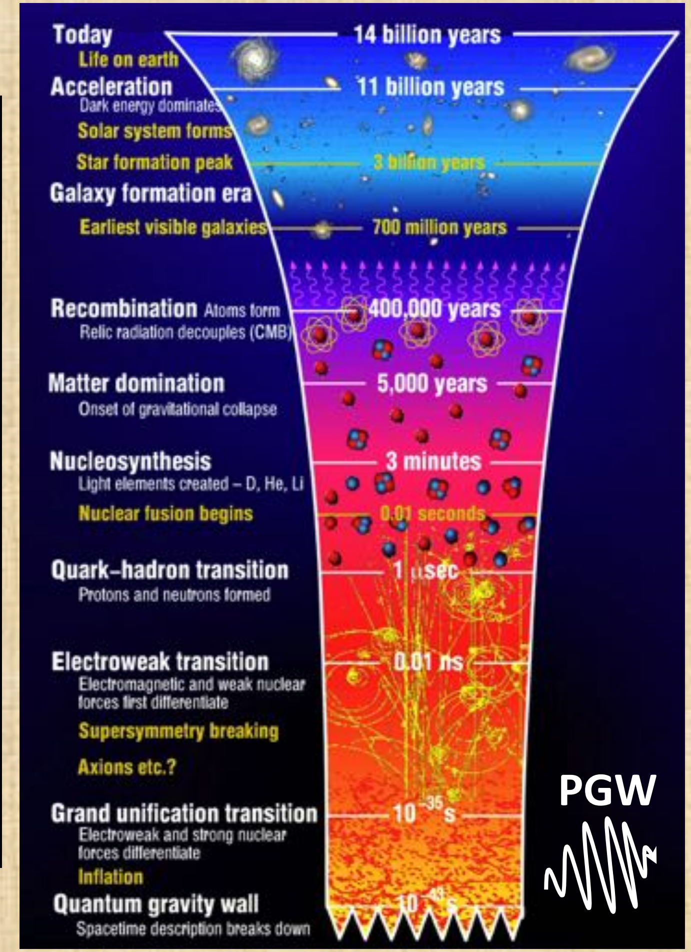
(Received 21 January 2016; published 11 February 2016)



In the source frame, the initial black hole masses are $36^{+5}_{-4} M_\odot$ and $29^{+4}_{-4} M_\odot$, and the final black hole mass is $62^{+4}_{-4} M_\odot$, with $3.0^{+0.5}_{-0.5} M_\odot c^2$ radiated in gravitational waves. All uncertainties define 90% credible intervals.

Primordial gravitational waves





$T_{Planck} = 10^{19}$ Gev $= 10^{32}$ K

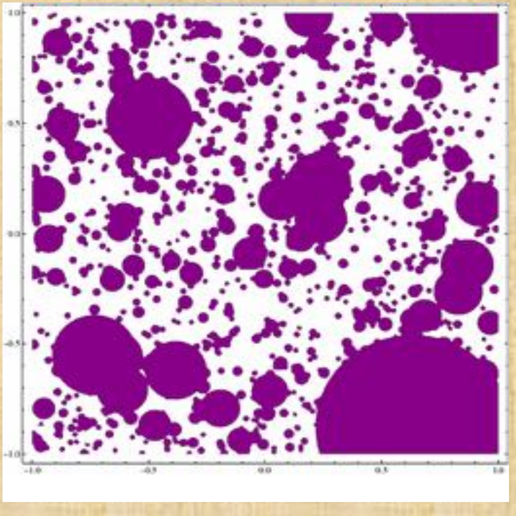
$t_{Planck} = 10^{-43}$ sec

$L_{Planck} = 10^{-35}$ m

Sources of primordial gravitational waves

✓ First-order phase transition: **vacuum bubble collisions**

➔ metric perturbations of **order one**



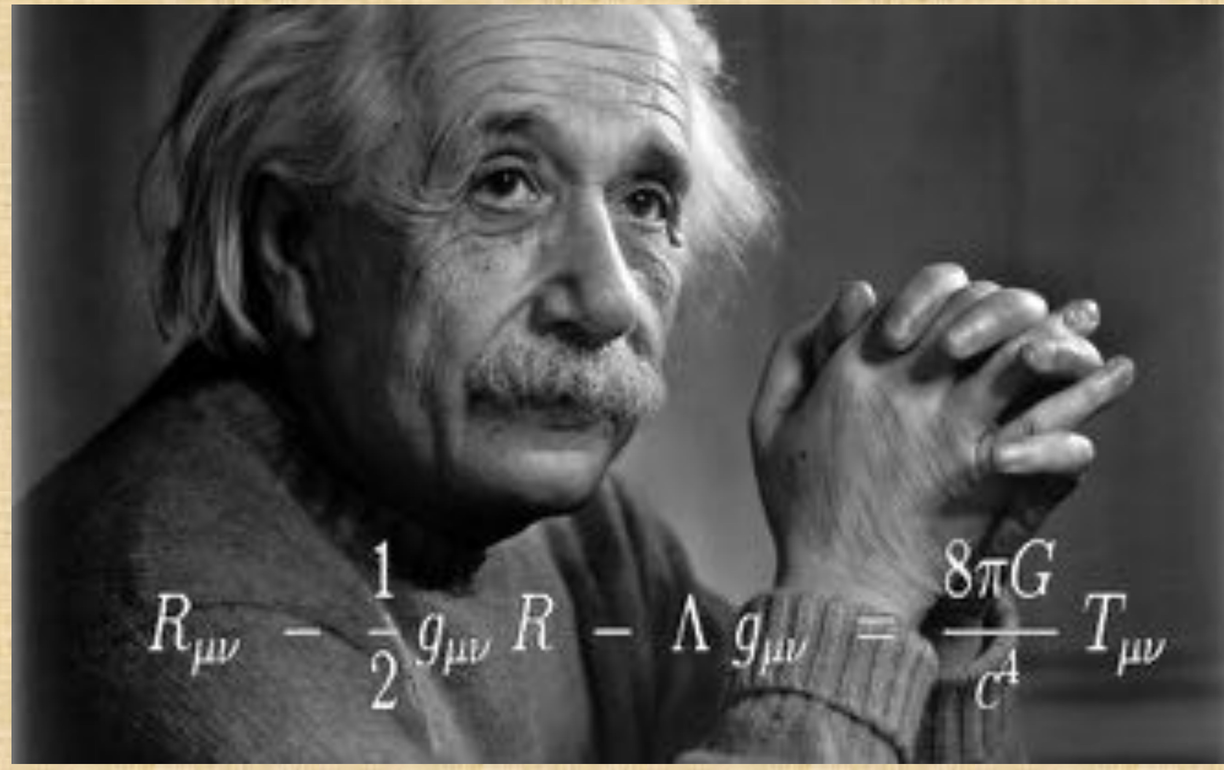
[Turner & Wilczek, PRL, 1990; Kosowsky et al., PRL, 1992]
[Reviews: Kleban, Class. Quantum Grav., 2011;
Binétruy et al., J. Cosm. Astropart. Phys., 2012
Guzzetti et al., Riv. Nuovo Cimento, 2016]

$$T_* = 10^{15}\ \text{Gev} = 10^{28}\ \text{K}, \quad t_* = 10^{-36}\ \text{sec} \quad \Rightarrow \quad h \approx 0.3$$

✓ Cosmic strings

✓ Inflation ($t = 10^{-35}$ sec to 10^{-32} sec)….

Einstein equation



10 **nonlinear** partial differential equations

$R_{\mu\nu}$: Ricci tensor
R: curvature
$g_{\mu\nu}$: metric tensor
Λ: cosmological constant

$T_{\mu\nu}$: stress-energy
$G = 6.67\ 10^{-11}\ m^3 kg^{-1} s^{-2}$
$c = 2.99\ 10^8\ ms^{-1}$

Friedmann-Lemaître solutions

Hypothesis: the Universe is homogeneous and isotropic

$ds^2 = dt^2 - a(t)^2\, d\ell^2$ a(t): cosmic scale factor

Examples:

$\Lambda=0$ and no curvature => $a(t) \approx t^{2/3}$ (Einstein-de Sitter Universe)

$\Lambda>0$ and no curvature => $a(t) \approx \exp[(\Lambda/3)^{1/2}\, ct]$ **Inflation** model

What is the meaning of Λ ? **DARK ENERGY,** vacuum energy, scalar field…

"So far, the details of inflation are unknown, and the whole idea of inflation remains a speculation, though one that is increasingly plausible." Weinberg, Cosmology, 2008.

Why do we need inflation ?

Inflation theory was developed in the early 1980s to explain the large-scale structure of the Universe [eg. Guth, 1981]

✓ The cosmic microwave background (CMB) is statistically **uniform**

➔ **horizon** (causal) problem

✓ The Universe is apparently **flat**

➔ **flatness** problem

A rapid – superluminal – expansion can explain these properties

Recurrent criticism: the inflation fields are still unknown…

➔ it is still considered as an **unsolved problem**

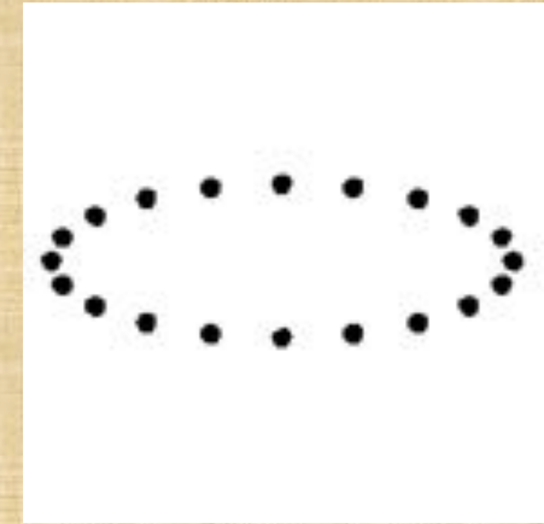
Gravitational waves

Exact linear solutions in an empty – flat – Universe:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \text{ where } h_{\mu\nu} \ll 1$$

$$R_{\mu\nu} = 0$$

The effect of a + gravitational wave on a ring of particles (h ≈ 0.5)



$$\omega_{\mathbf{k}} = c|\mathbf{k}| = ck$$

'Incompressible' waves

$$h^{+}_{\mu\nu} = a \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$h^{\times}_{\mu\nu} = b \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Weakly nonlinear general relativity

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \text{ where } h_{\mu\nu} \ll 1$$ $$R_{\mu\nu} = 0$$ Empty Universe

$$R_{\mu\nu} = R_{\mu\nu}^{(1)} + R_{\mu\nu}^{(2)} \text{ with } R_{\mu\nu}^{(1)} = -\Box h_{\mu\nu} \text{ and}$$

$$\begin{aligned} R_{\mu\nu}^{(2)} = &+\frac{1}{4}\left[2\frac{\partial h_\sigma^\alpha}{\partial x^\alpha} - \frac{\partial h_\alpha^\alpha}{\partial x^\sigma}\right]\left[\frac{\partial h_\mu^\sigma}{\partial x^\nu} + \frac{\partial h_\nu^\sigma}{\partial x^\mu} - \frac{\partial h_{\mu\nu}}{\partial x_\sigma}\right] \\ &-\frac{1}{2}h^{\lambda\alpha}\left[\frac{\partial^2 h_{\lambda\alpha}}{\partial x^\nu \partial x^\mu} - \frac{\partial^2 h_{\mu\alpha}}{\partial x^\nu \partial x^\lambda} - \frac{\partial^2 h_{\lambda\nu}}{\partial x^\alpha \partial x^\mu} + \frac{\partial^2 h_{\mu\nu}}{\partial x^\alpha \partial x^\lambda}\right] \\ &-\frac{1}{4}\left[\frac{\partial h_{\sigma\nu}}{\partial x^\lambda} + \frac{\partial h_{\sigma\lambda}}{\partial x^\kappa} - \frac{\partial h_{\lambda\nu}}{\partial x^\sigma}\right]\left[\frac{\partial h_\mu^\sigma}{\partial x_\lambda} + \frac{\partial h^{\sigma\lambda}}{\partial x^\mu} - \frac{\partial h_\mu^\lambda}{\partial x_\sigma}\right]. \end{aligned} \quad (1)$$

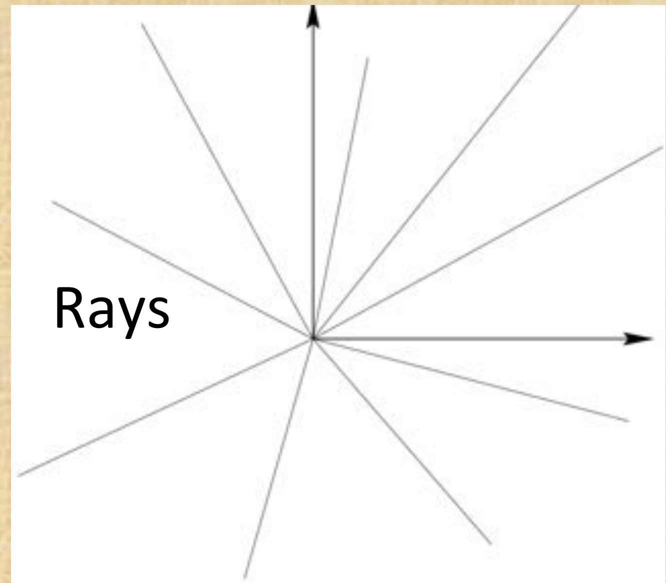
Resonance condition

For three-wave interactions:

$$\mathbf{k} = \mathbf{k}_1 + \mathbf{k}_2 \text{ and } \omega_{\mathbf{k}} = \omega_{\mathbf{k}_1} + \omega_{\mathbf{k}_2}$$

$$\omega_{\mathbf{k}} = c|\mathbf{k}| = ck$$

Similar to acoustic waves



After some calculations…

$$\Rightarrow \quad R^{(2)}_{\mu\nu} = 0$$

Three-wave interactions of weak GW turbulence are absent

Theory for four-wave interactions

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \text{ where } h_{\mu\nu} \ll 1$$

$$R_{\mu\nu} = R^{(1)}_{\mu\nu} + \cancel{R^{(2)}_{\mu\nu}} + R^{(3)}_{\mu\nu} + R^{(4)}_{\mu\nu} = 0$$

negligible

Empty Universe

Too heavy problem to be treated analytically => **need simplifications**

2.5 + 1 diagonal metric tensor

$$g_{\mu\nu} = \begin{pmatrix} -(H_0)^2 & 0 & 0 & 0 \\ 0 & (H_1)^2 & 0 & 0 \\ 0 & 0 & (H_2)^2 & 0 \\ 0 & 0 & 0 & (H_3)^2 \end{pmatrix}$$

$$H_0 = e^{-\lambda}\gamma, \;\; H_1 = e^{-\lambda}\beta, \;\; H_2 = e^{-\lambda}\alpha, \;\; H_3 = e^{\lambda}$$

[Hadad & Zakharov, 2014]

- Self-consistent system; we need to solve:

$$R_{01} = R_{02} = R_{12} = 0 \text{ and } R_{\mu\mu} = 0$$

7 equations can be reduced to **4** equations [Hadad & Zakharov, 2014]

- In the linear approximation: $\alpha = \beta = \gamma = 1$

=> $$\ddot{\lambda} - \partial_{xx}\lambda - \partial_{yy}\lambda = 0$$

$$\lambda = c_1 \exp(-i\omega_{\mathbf{k}}t + i\mathbf{k}\cdot\mathbf{x}) + c_2 \exp(i\omega_{\mathbf{k}}t + i\mathbf{k}\cdot\mathbf{x})$$

where $\mathbf{k} = (p, q)$ is a 2D wave vector

'+ GW' only

Hamiltonian formalism

Lagrangian density of the Einstein-Hilbert action

$$\mathcal{L} = \frac{1}{2}\left[\frac{\alpha\beta}{\gamma}\dot{\lambda}^2 - \frac{\alpha\gamma}{\beta}(\partial_x\lambda)^2 - \frac{\beta\gamma}{\alpha}(\partial_y\lambda)^2 - \frac{\dot{\alpha}\dot{\beta}}{\gamma} + \frac{(\partial_x\alpha)(\partial_x\gamma)}{\beta} + \frac{(\partial_y\beta)(\partial_y\gamma)}{\alpha}\right]$$

[Hadad & Zakharov, 2014]

Normal variables

$$\lambda_{\mathbf{k}} = \frac{a_{\mathbf{k}} + a^*_{-\mathbf{k}}}{\sqrt{2k}}, \quad \dot{\lambda}_{\mathbf{k}} = \frac{\sqrt{k}(a_{\mathbf{k}} - a^*_{-\mathbf{k}})}{i\sqrt{2}},$$

$$i\dot{a}_{\mathbf{k}} = \frac{\partial H}{\partial a^*_{\mathbf{k}}} \quad \text{where} \quad H = H_{\text{free}} + H_{\text{int}}$$

$$H_{\text{free}} = \sum_{\mathbf{k}} k|a_{\mathbf{k}}|^2$$

$$H_{\text{int}} = \frac{1}{2}\sum_{1,2,3}\delta_{123}\Big\{(-\tilde{\alpha}_1 - \tilde{\beta}_1 + \tilde{\gamma}_1)\dot{\lambda}_2\dot{\lambda}_3 - \left[(\tilde{\alpha}_1 - \tilde{\beta}_1 + \tilde{\gamma}_1)p_2p_3 + (-\tilde{\alpha}_1 + \tilde{\beta}_1 + \tilde{\gamma}_1)q_2q_3\right]\lambda_2\lambda_3\Big\} + \frac{1}{2}\sum_{\mathbf{k}}\left[\dot{\alpha}_{\mathbf{k}}\dot{\beta}^*_{\mathbf{k}} - (p^2\alpha_{\mathbf{k}} + q^2\beta_{\mathbf{k}})\gamma^*_{\mathbf{k}}\right], \qquad (12)$$

Weak turbulence formalism

$$n_{\mathbf{k}} = \lim_{L\to\infty} \frac{L^2}{4\pi^2}\langle |a_{\mathbf{k}}|^2\rangle,$$ Energy spectrum

$$\dot{n}_{\mathbf{k}} = 4\pi \int |T_{\mathbf{k}_1\mathbf{k}_2}^{\mathbf{k}\mathbf{k}_3}|^2\, n_{\mathbf{k}_1} n_{\mathbf{k}_2} n_{\mathbf{k}_3} n_{\mathbf{k}} \left[\frac{1}{n_{\mathbf{k}}} + \frac{1}{n_{\mathbf{k}_3}} - \frac{1}{n_{\mathbf{k}_1}} - \frac{1}{n_{\mathbf{k}_2}}\right] \delta(\mathbf{k}+\mathbf{k}_3-\mathbf{k}_1-\mathbf{k}_2)\,\delta(\omega_{\mathbf{k}}+\omega_{\mathbf{k}_3}-\omega_{\mathbf{k}_1}-\omega_{\mathbf{k}_2})\, d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3,$$

with $T_{34}^{12} = \frac{1}{4}(W_{34}^{12}+W_{34}^{21}+W_{43}^{12}+W_{43}^{21}),\ W_{34}^{12} = Q_{34}^{12}+Q_{12}^{34}$ $\qquad H_{3\to 1} = 0$

$$Q_{34}^{12} = \frac{1}{4\sqrt{k_1k_2k_3k_4}}\left\{2\left(\frac{p_4}{p_1-p_3}-\frac{q_4}{q_1-q_3}\right)\frac{k_2(p_1p_3-q_1q_3)}{k_1-k_3} - 2\left(\frac{p_4}{p_1-p_3}+\frac{q_4}{q_1-q_3}\right)\frac{k_1k_2k_3}{k_1-k_3}\right. \tag{14}$$
$$\left. +\left(\frac{p_2}{p_1+p_2}-\frac{q_2}{q_1+q_2}\right)\frac{k_1(p_3p_4-q_3q_4)}{k_1+k_2} - \left(\frac{p_2}{p_1+p_2}+\frac{q_2}{q_1+q_2}\right)\frac{k_1k_3k_4}{k_1+k_2} + \frac{2k_1k_3p_2q_4}{(p_1+p_2)(q_1+q_2)} + \frac{2k_1p_3(q_2k_4+k_2q_4)}{(p_1-p_3)(q_1-q_3)}\right\}.$$

In our scenario, the early Universe is governed by this kinetic equation

Properties of the kinetic equation

$$\mathcal{E} = \int \omega_{\mathbf{k}} n_{\mathbf{k}} d\mathbf{k} = \text{const},$$

$$\mathcal{N} = \int n_{\mathbf{k}} d\mathbf{k} = \text{const}.$$

Isotropic constant-flux stationary Kolmogorov-Zakharov solutions:

$$n_{\mathbf{k}} \sim k^{-2} \quad \text{and} \quad n_{\mathbf{k}} \sim k^{-5/3}$$

Direct cascade of energy

Inverse cascade of wave action

Phenomenology of GW turbulence

Kinetic equation: $\partial_t n_k = \cancel{\epsilon^2()} + \epsilon^4() + \ldots$ $\quad \epsilon = \frac{\tau_{GW}}{\tau_{NL}}$ << 1

$$\Rightarrow \quad \tau_{cascade} \sim \frac{1}{\epsilon^4}\tau_{GW} \sim \left(\frac{\tau_{NL}}{\tau_{GW}}\right)^3 \tau_{NL}$$

$$\tau_{GW} \sim 1/\omega$$

$$\tau_{NL} \sim \ell/(hc)$$

Energy density: $E \sim \frac{c^4}{32\pi G}\frac{h^2}{\ell^2} \sim \omega\, N$

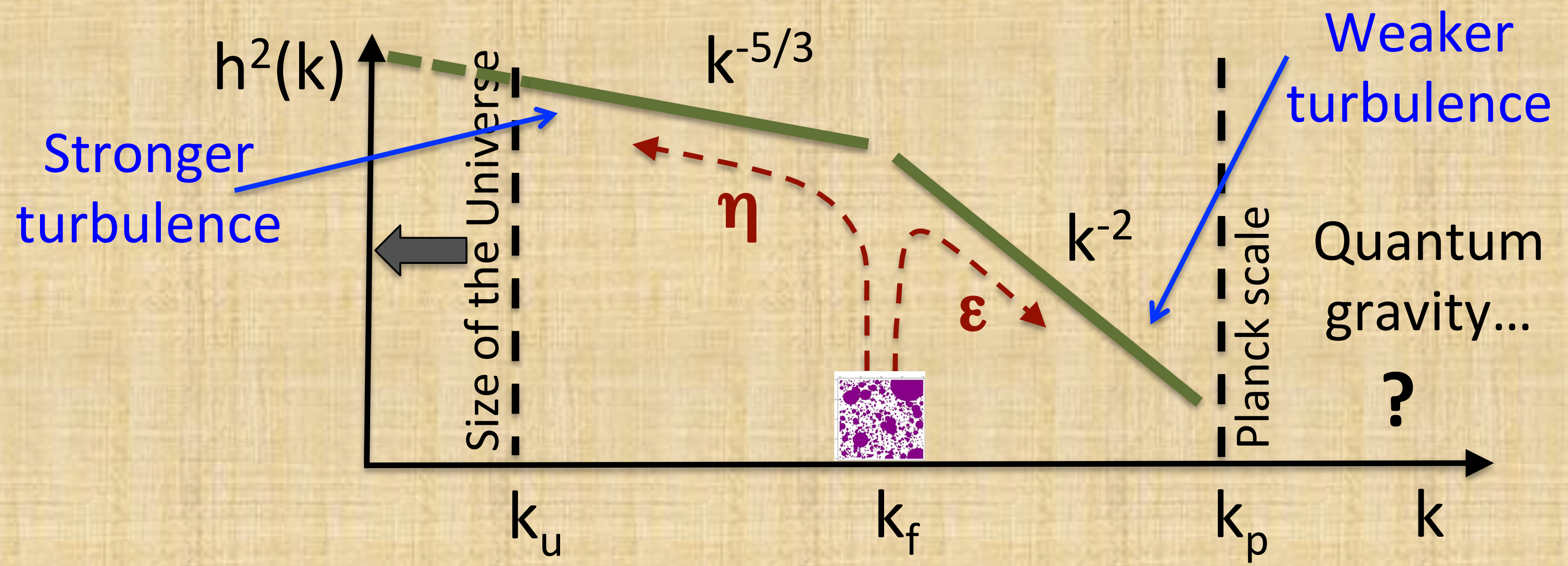
[Maggiore, 2008]

1D energy and wave action spectra:

$$\varepsilon \sim \frac{E}{\left(\frac{\tau_{NL}}{\tau_{GW}}\right)^3 \tau_{NL}} \sim \frac{E}{\left(\frac{\ell}{h}\right)^4 \omega^3} \sim \frac{E^3}{k^3} \sim E_k^3 \quad \Rightarrow \quad \boxed{E_k \sim \varepsilon^{1/3}}$$

$$\eta \sim \frac{N}{\left(\frac{\tau_{NL}}{\tau_{GW}}\right)^3 \tau_{NL}} \sim \frac{N}{\left(\frac{\ell}{h}\right)^4 \omega^3} \sim \frac{N^3}{k} \sim N_k^3 k^2 \quad \Rightarrow \quad \boxed{N_k \sim \eta^{1/3} k^{-2/3}}$$

Fluctuating space-time / GW spectrum



Direct / inverse cascade:

$$\varepsilon_f = \varepsilon_u + \varepsilon_p$$

$$\eta_f = \eta_u + \eta_p$$

$$\varepsilon = k\eta$$

$$\varepsilon_u = k_u \eta_u$$

$$\varepsilon_p = k_p \eta_p$$

$$\Rightarrow$$

$$\frac{\varepsilon_p}{\varepsilon_u} = \frac{k_p}{k_u}\frac{1 - k_u/k_f}{k_p/k_f - 1} \sim \frac{k_f}{k_u} \to +\infty$$ **direct cascade**

$$\frac{\eta_p}{\eta_u} = \frac{1 - k_u/k_f}{k_p/k_f - 1} \sim \frac{k_f}{k_p} \to 0$$ **inverse cascade**

if $k_u \ll k_f \ll k_p$

Properties of the KE

$$\mathcal{E} = \int \omega_{\mathbf{k}} n_{\mathbf{k}} d\mathbf{k} = \text{const},$$

$$\mathcal{N} = \int n_{\mathbf{k}} d\mathbf{k} = \text{const}.$$

Isotropic constant-flux stationary Kolmogorov-Zakharov solutions:

$$n_{\mathbf{k}} \sim k^{-2} \quad \text{and} \quad n_{\mathbf{k}} \sim k^{-5/3}$$

Direct cascade of energy

Inverse cascade of wave action

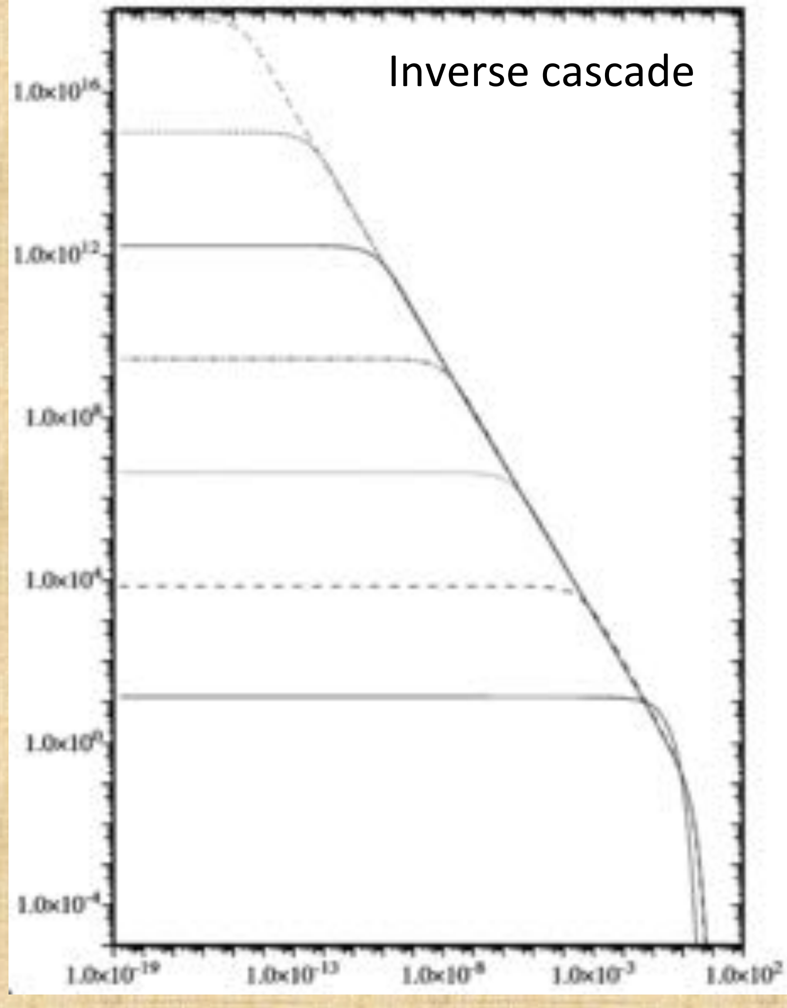
Spectrum with **finite capacity**

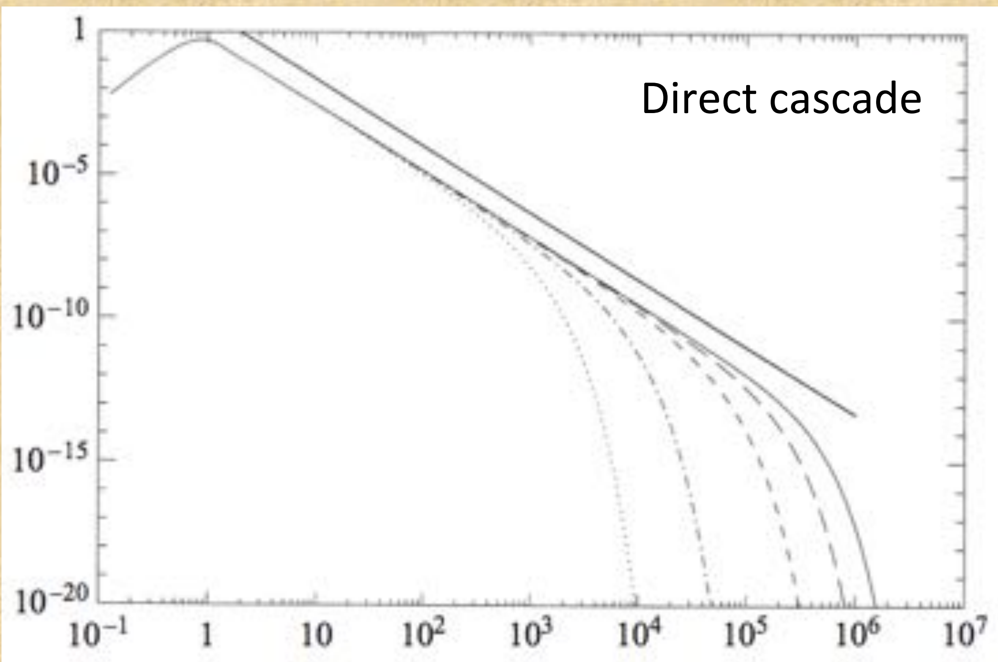
The inverse cascade from k_f to k=0 happens in a **finite** time !

Explosive mechanism !

Spectra with finite capacity

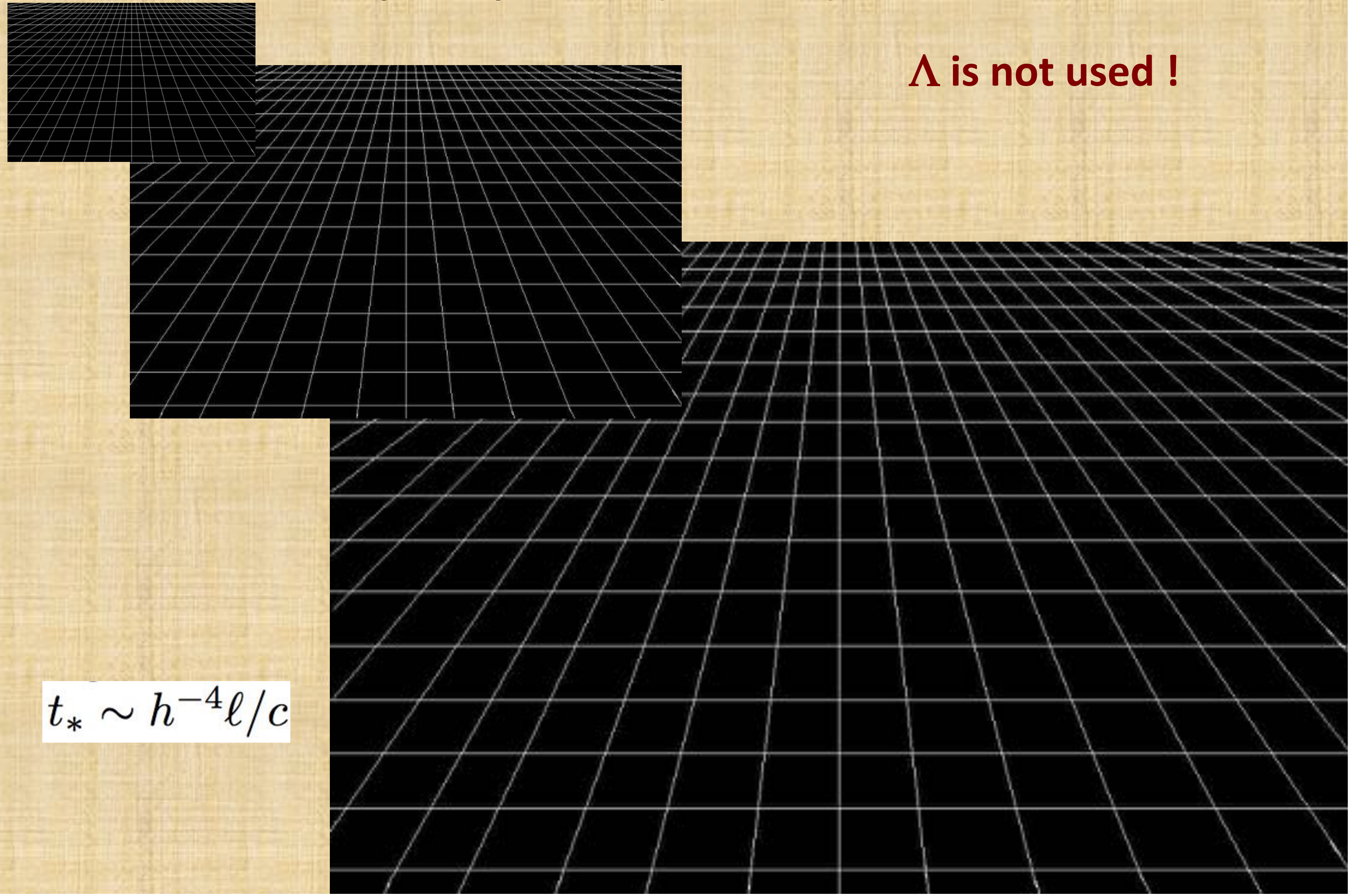
Bose-Einstein condensate
[Lacaze et al., 2001]





Alfvén waves
[SG et al., 2000]

Turbulence provides a mechanism of rapid expansion (inflation) of the Universe



Λ is not used !

$$t_* \sim h^{-4}\ell/c$$

Conclusion

Galtier & Nazarenko, submitted; arXiv:1703.09069v1

- Derivation of the weak GW turbulence equations
- 4-wave interactions are needed
- Inverse cascade with finite capacity – spectrum $h^2(k) \sim k^{-5/3}$
- Turbulence provides a scenario for the cosmological inflation
- The cosmological constant Λ is not introduced

Dark energy = Turbulence

Can we observe primordial GW ?

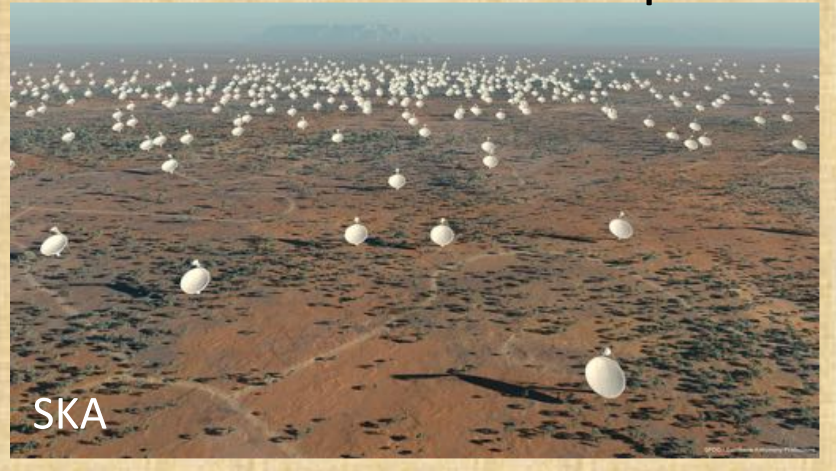
[Lasky et al., PRX, 2016]

Pulsar Timing Array (PTA):

Observations of a large sample of ms pulsars are combined to detect a slow (over several years) modulation of the signal

- ➢ Parkes PTA / Australia; 64m radio-telescope
- ➢ Square Kilometer Array (SKA) /Australia + South Africa
- ➢ Other (Europe, North America)

> 1000 radio-telescopes



The GW landscape by Janssen et al. (2014)