An Introduction to 2D Turbulence

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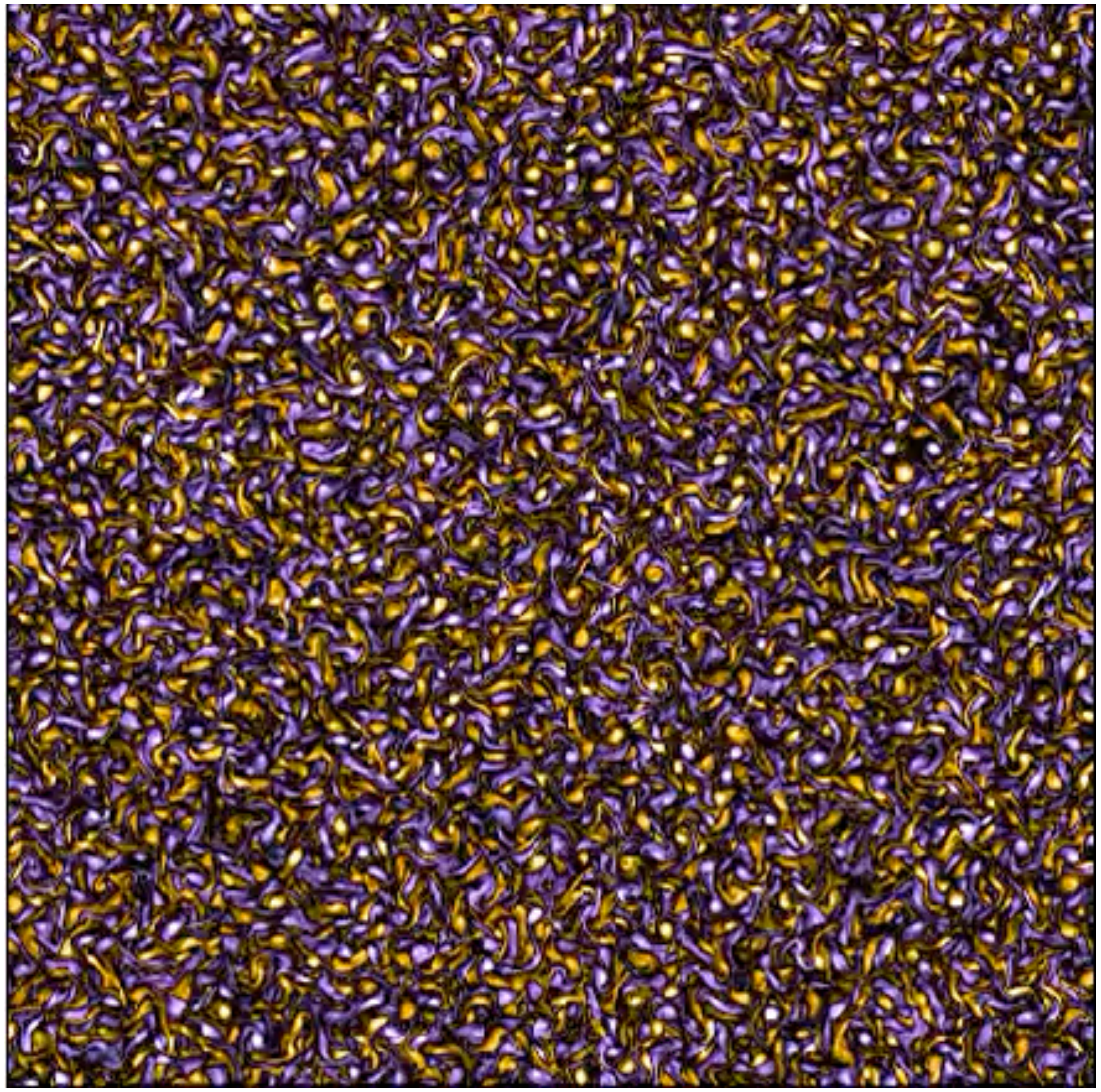

Outline

- Key results of 3D turbulence
- 2D turbulence theory
- Experiments and simulations
- Energy condensation and mean flows
- Thin layer turbulence: 2D to 3D transition

Waves, coherent structures, and turbulence, UEA, 30th October 2019

Why is 2D turbulence important?

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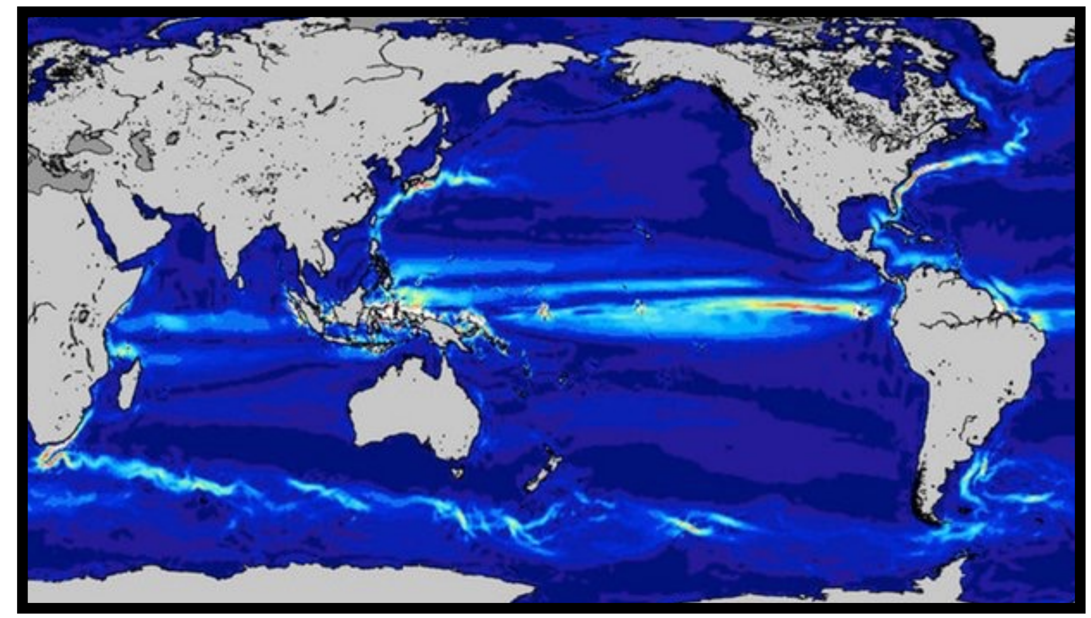
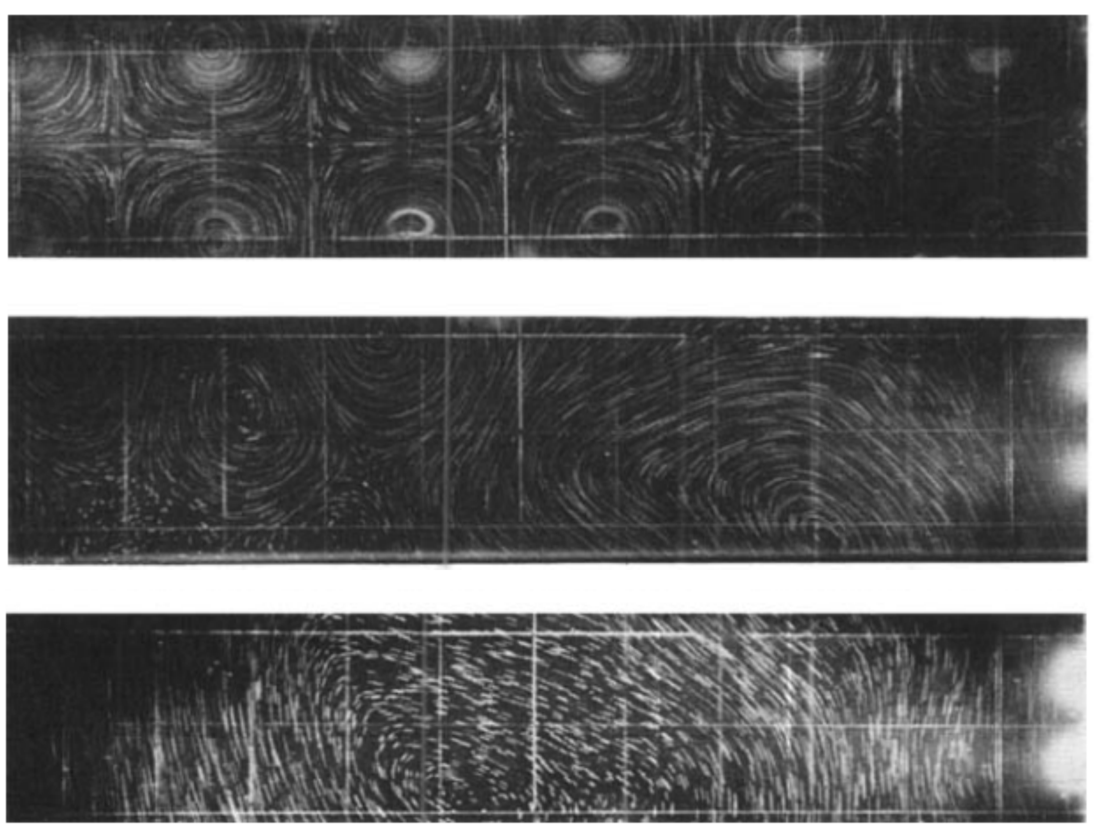
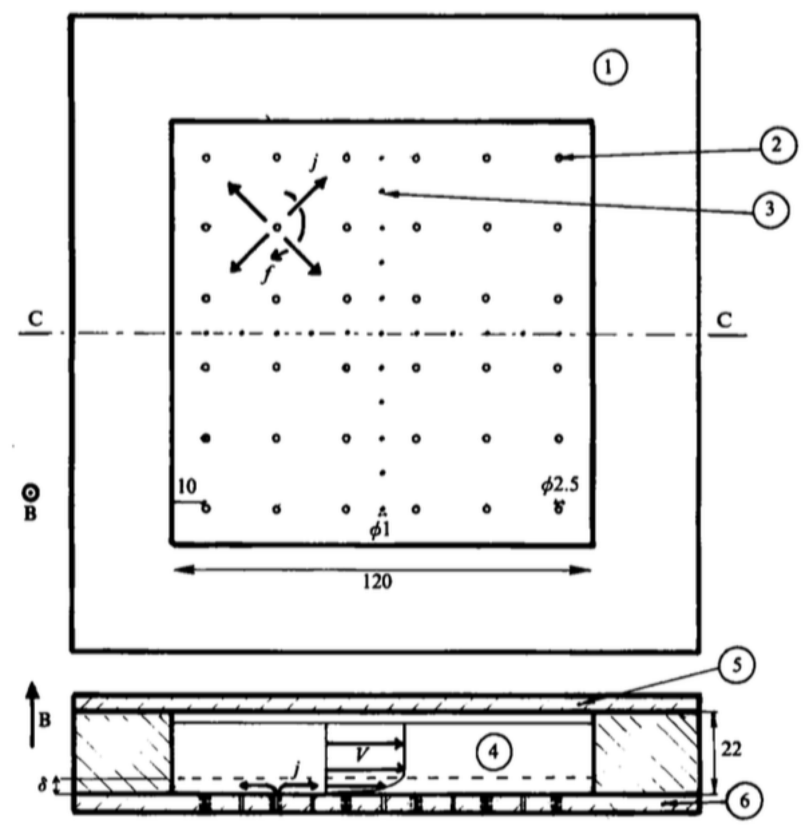
Mean flows in geophysical turbulence



Thin-layer fluid experiments

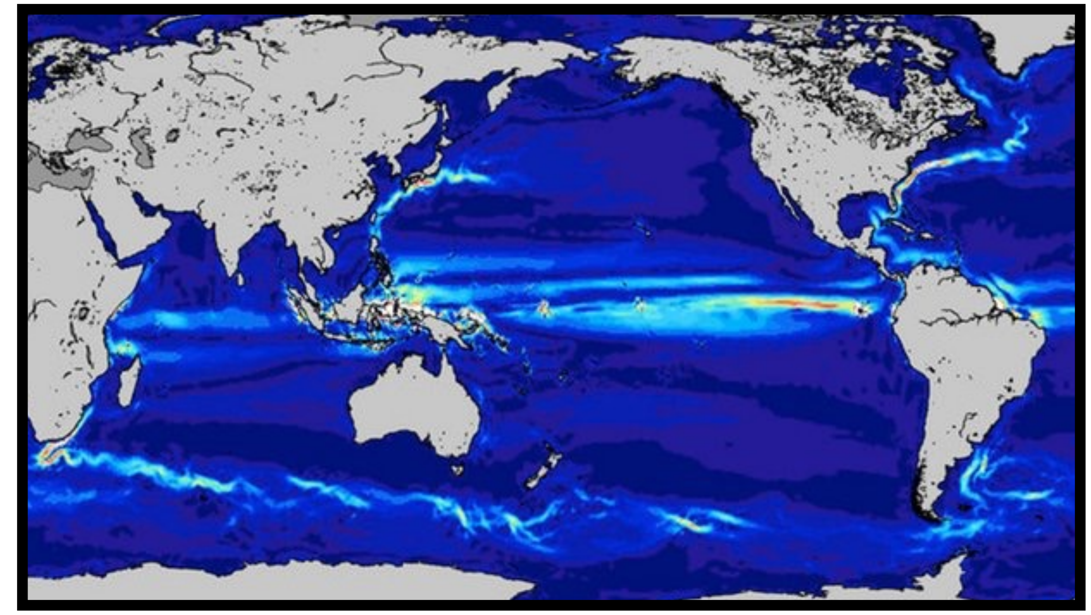




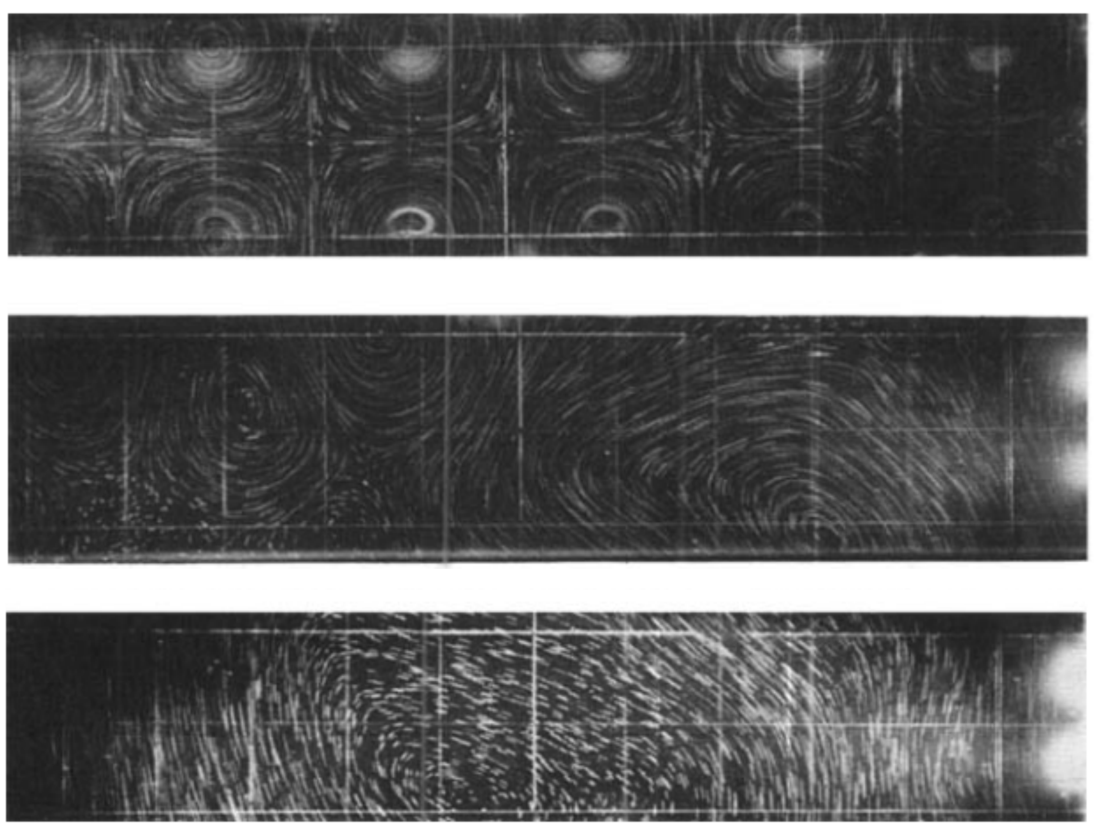
Sommeria,
J. Fluid Mech. **170**,
139, (1986)

Why is 2D turbulence important?

Mean flows in geophysical turbulence



Thin-layer fluid experiments



Properties

- Stable large-scale coherent mean flow
- Typically generated out of small-scale fluctuations
- In non-equilibrium balance between forcing and dissipation

Navier-Stokes and turbulence

3D Navier-Stokes equations

Claude-Louis Navier and George Stokes (1827-1845)

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\,\mathbf{v} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{v} + \mathbf{f}$$

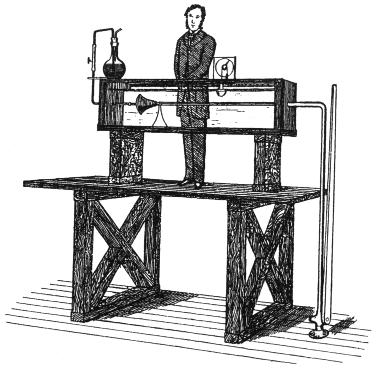
$$\nabla\cdot\mathbf{v} = 0$$

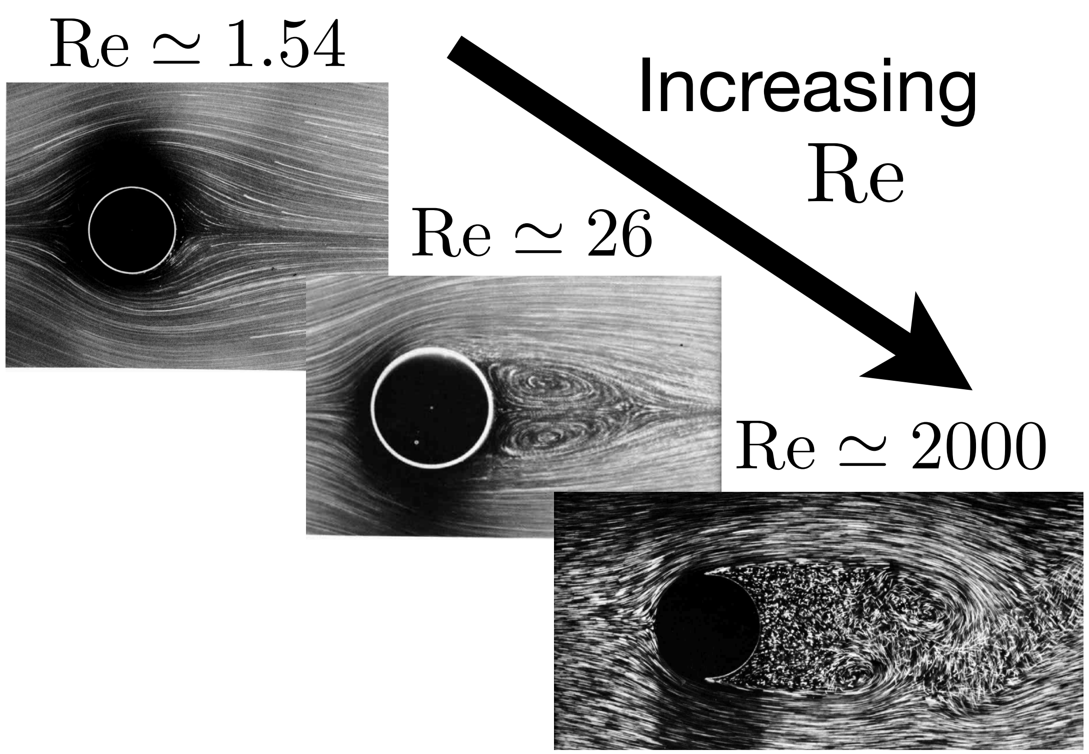
Reynolds number

Osbourne Reynolds (1883)

- Ratio of the nonlinear advection term to that of viscous diffusion

$$\mathrm{Re} = \frac{|(\mathbf{v}\cdot\nabla)\mathbf{v}|}{|\nu\nabla^2\mathbf{v}|} \sim \frac{VL}{\nu}$$





Turbulence appears when $1 \ll \mathrm{Re}$

The Navier-Stokes equation

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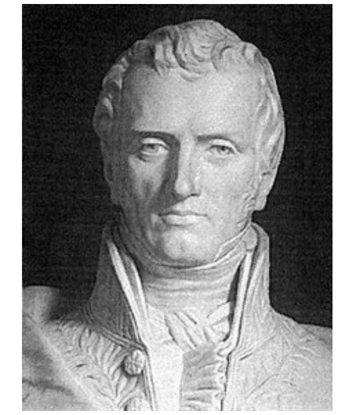
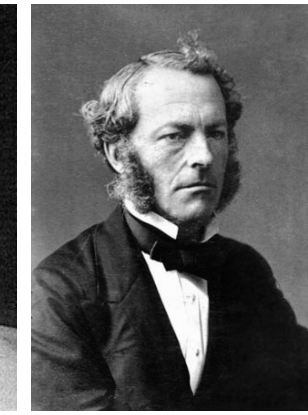
3D Navier-Stokes equations

Claude-Louis Navier and George Stokes (1827-1845)





$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\,\mathbf{v} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{v} + \mathbf{f}$$

$$\nabla\cdot\mathbf{v} = 0$$

The energy balance equation

$$\boxed{\frac{dE}{dt} = -2\nu Z = -\epsilon}$$

Kinetic Energy $$E = \frac{1}{2}\int |\mathbf{v}|^2\, d\mathbf{x}$$

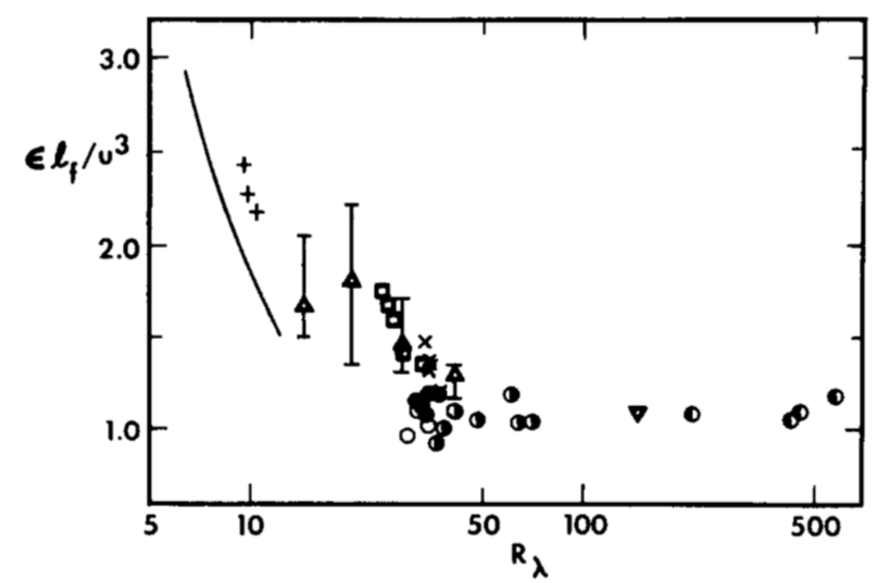
Enstrophy $$Z = \frac{1}{2}\int |\nabla\times\mathbf{v}|^2\, d\mathbf{x}$$

- In the inviscid limit, the Navier-Stokes equation conserves the kinetic energy

Dissipative anomaly

- However, in the inviscid limit, experimentally the energy dissipation rate ϵ remains finite

$$\epsilon = \lim_{\nu\to 0} 2\nu Z > 0$$



Kolmogorov’s theory for turbulence

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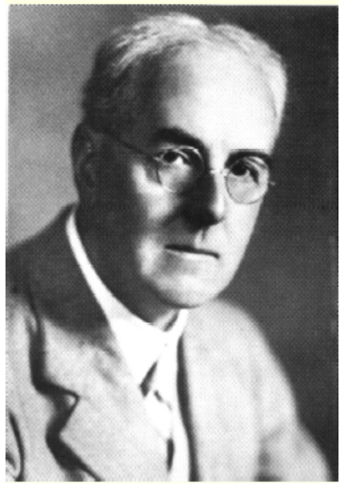
Richardson’s cascade picture



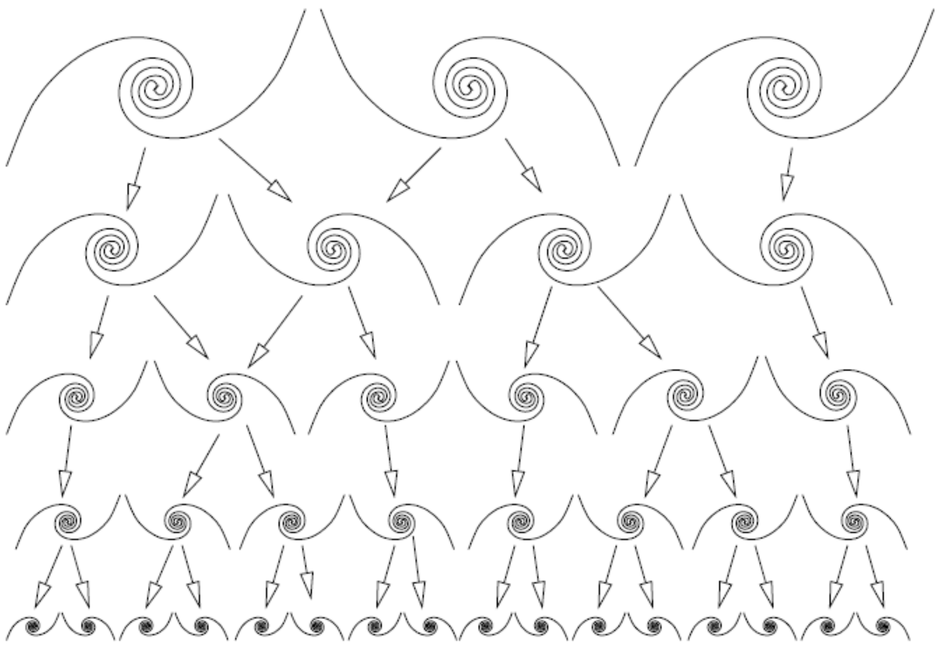
“Big whirls have little whirls that feed on their velocity, and little whirls have lesser whirls and so on to viscosity.”

Lewis Fry Richardson, Weather Prediction by Numerical Process, (1922)

Kolmogorov’s four-fifths law

- **Inertial range dynamics**: scale separation between forcing and dissipation
- Under the assumptions of **scale invariance, isotropy**, and **homogeneity** Kolmogorov proved that for the velocity increments $\delta v_r(\mathbf{x}) = [\mathbf{v}(\mathbf{x}+\mathbf{r}) - \mathbf{v}(\mathbf{x})] \cdot \mathbf{r}$

$$\boxed{\langle (\delta v_r)^3 \rangle = -\frac{4}{5}\epsilon r}$$



A. N. Kolmogorov, Dokl. Akad. Nauk SSSR **30**, 299 (1941)

Kolmogorov’s energy spectrum

Kolmogorov’s Energy Spectrum

- Distribution of energy in scale-space can be observe by computing the 1D energy spectrum E_k defined through

$$E = \frac{1}{2}\int_{\mathcal{D}} |\mathbf{v}|^2 \, d\mathbf{r} = \int E_k \, dk$$

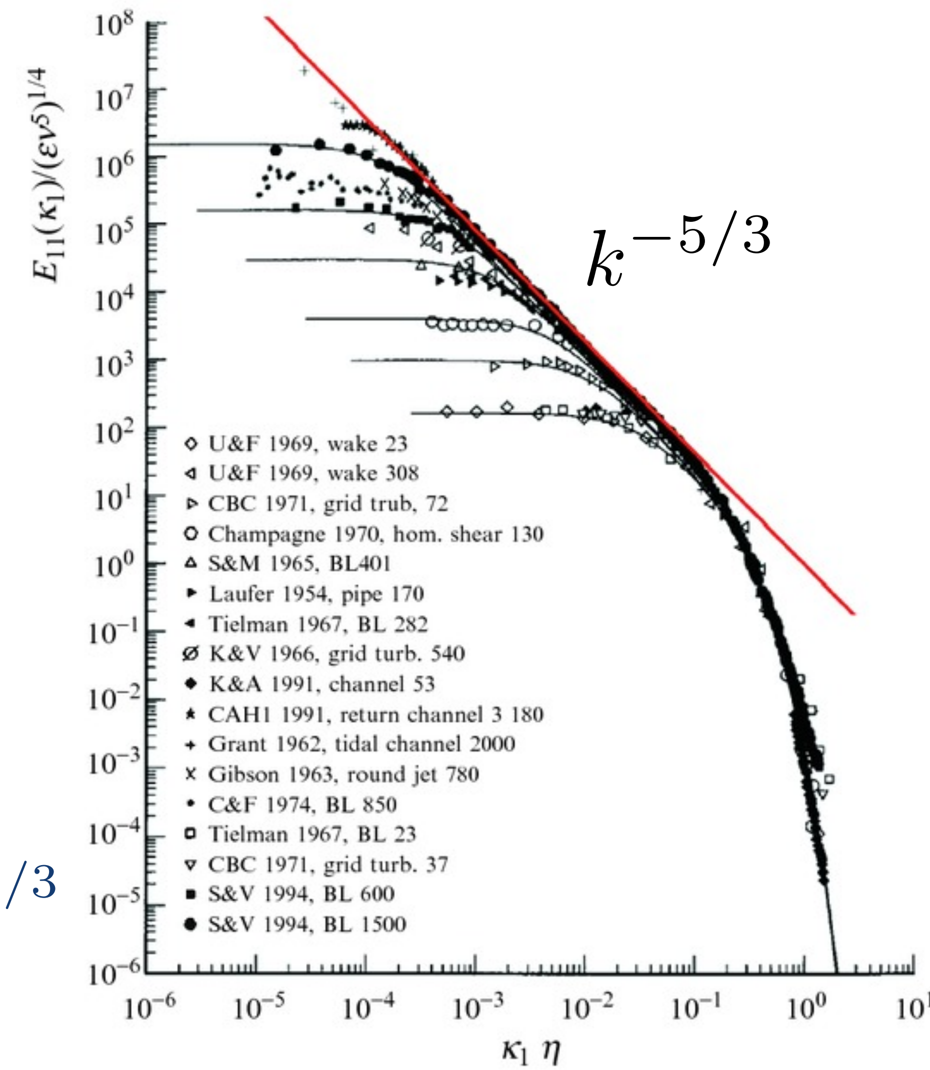
- Assuming we satisfy Kolmogorov’s four-fifths law

$$\langle (\delta v_r)^3 \rangle = -\frac{4}{5}\epsilon r$$

- Then, by dimensional arguments the **second order velocity increment correlator scales** as

$$\boxed{\langle (\delta v_r)^2 \rangle \propto \epsilon^{2/3} r^{2/3} \quad \Rightarrow \quad E_k \propto \epsilon^{2/3} k^{-5/3}}$$

A. N. Kolmogorov, Dokl. Akad. Nauk SSSR **30**, 299 (1941)
A. M Obukhov, Dokl. Akad. Nauk SSSR, **5**, 453–466, (1941)



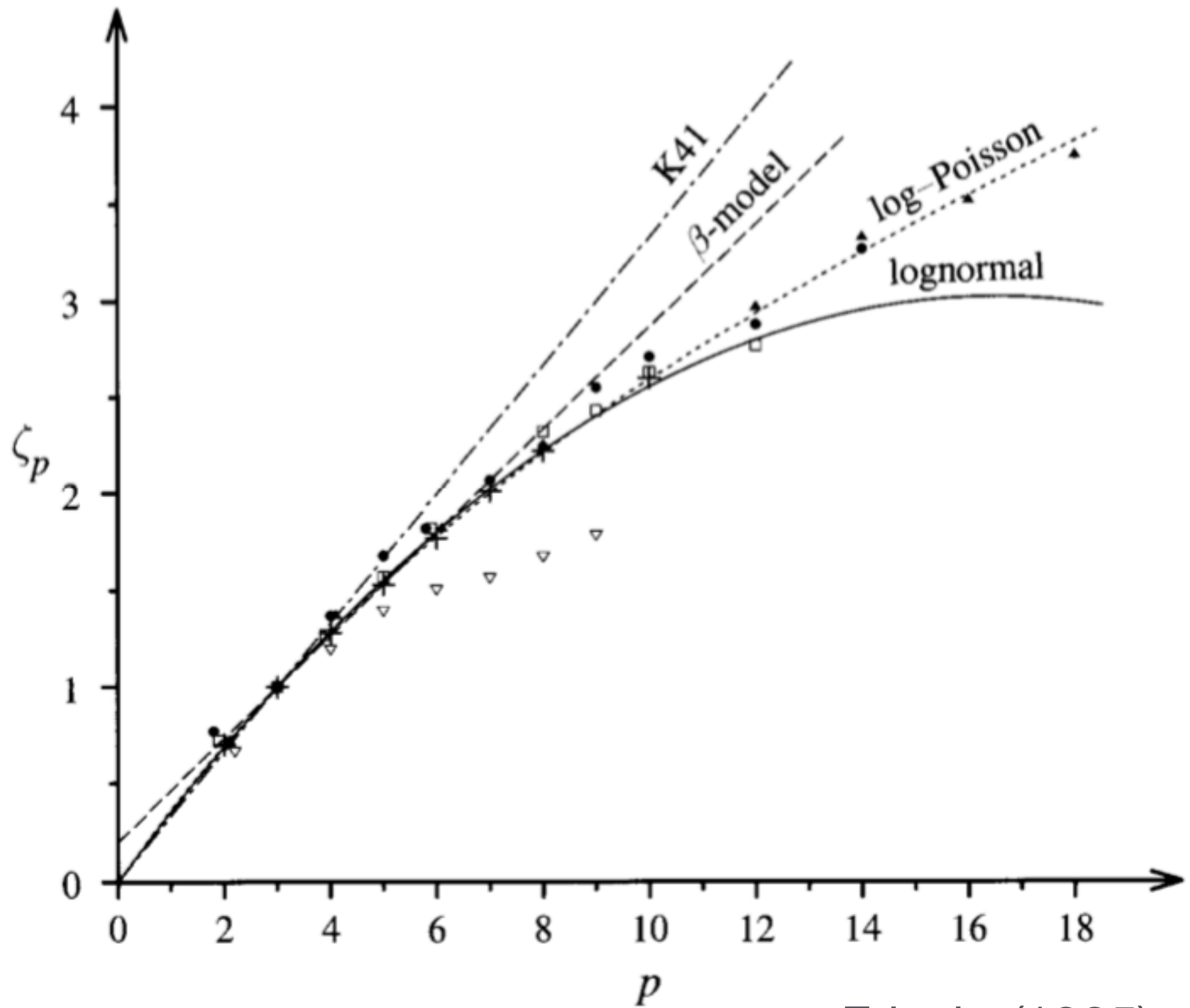
- A ‘universal’ dimensionless prefactor for $E_k = C\epsilon^{2/3}k^{-5/3}$ is experimentally measure to be around $C \simeq 1.5$

Turbulence intermittency

Structure function scaling from K41

- From the **assumptions of Kolmogorov K41 theory**, we expect that the statistical moments of turbulent velocity structure functions should scale as

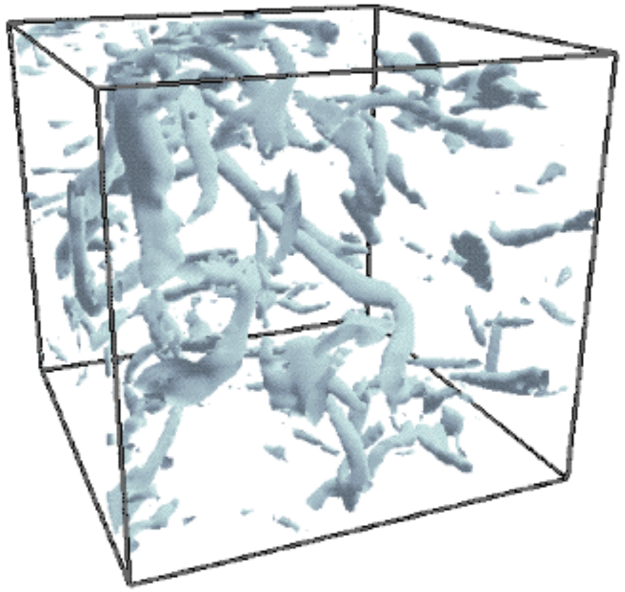
$$\boxed{\langle(\delta v_r)^p\rangle = C_n\,(\epsilon r)^{p/3}}$$



Frisch, (1995)

Intermittency

- **Strong deviations** from K41 for $p > 3$
- Several models have been proposed to explain the deviation from $\zeta_p = p/3$
- Intermittency is related to velocity phase correlations

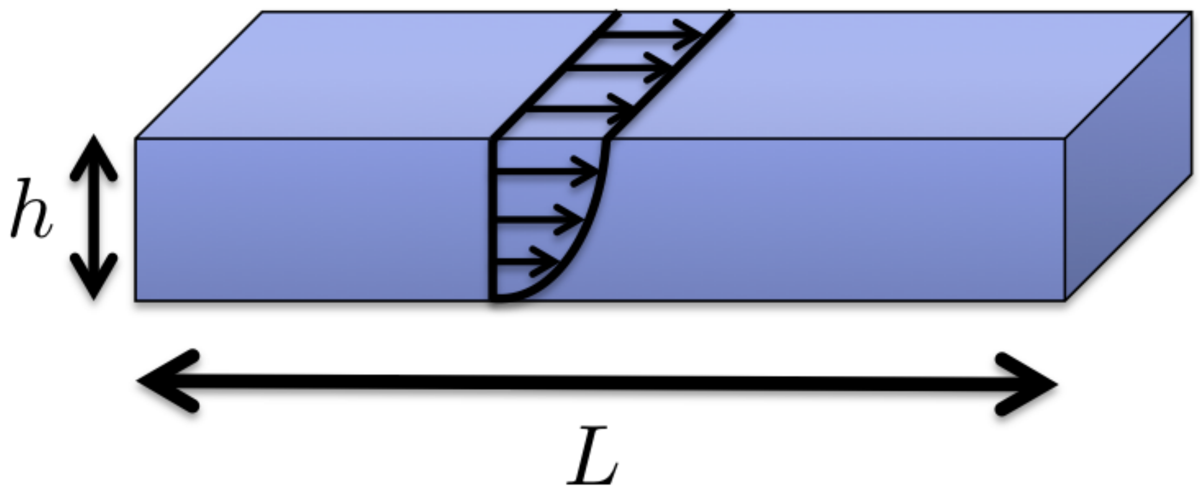


Leveque and She, (1993)

2D Turbulence Theory and Experiments

Thin-layer fluid flows

Consider a thin-layer fluid flow governed by 3D Navier-Stokes equations



$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\,\mathbf{v} = -\nabla p + \nu \nabla^2 \mathbf{v} + \mathbf{f}$$

$$\nabla \cdot \mathbf{v} = 0 \qquad \mathbf{v} = (u, v, w)$$

- Neglect vertical motions because $w \sim O(h/L)(u, v)$
- Assume a Poiseuille velocity profile in the vertical direction $u(z), v(z) \propto z^2$

$$\nu \nabla^2_{(3D)} \mathbf{v} \quad \rightarrow \quad \nu \nabla^2_{(2D)} \mathbf{v} - \alpha \mathbf{v} \qquad\qquad \alpha \sim O(\nu/h^2)$$

Thin-layer approximation: 2D Navier-Stokes with linear friction

$$\boxed{\begin{array}{c} \dfrac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\,\mathbf{v} = -\nabla p + \nu \nabla^2 \mathbf{v} - \alpha \mathbf{v} + \mathbf{f} \\ \nabla \cdot \mathbf{v} = 0 \qquad\qquad \mathbf{v} = (u, v) \end{array}}$$

$$\alpha \left\{ \begin{array}{l} \text{Ekman friction (rotating flows)} \\ \text{Rayleigh friction (stratified flows)} \\ \text{Hartmann friction (MHD)} \\ \text{Air friction (soap films)} \end{array} \right.$$

The 2D Navier-Stokes equations

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The 2D Navier-Stokes is the **simplest turbulence model** for the large-scale motion of geophysical flows where rotation, stratification, or thin-layers suppresses vertical motions

Vorticity formalism of 2D Navier-Stokes

$$\frac{\partial \omega}{\partial t} + \mathbf{v} \cdot \nabla \omega = \nu \nabla^2 \omega - \alpha \omega + f_\omega$$

$$\omega = (\nabla \times \mathbf{v}) \cdot \mathbf{e}_z$$
$$\omega = \nabla^2 \psi$$
$$\mathbf{v} = \mathbf{e}_z \times \nabla \psi$$

2D Navier-Stokes has an infinite number of inviscid invariants

Energy

$$E = \frac{1}{2} \int_{\mathcal{D}} |\mathbf{v}|^2 \, d\mathbf{r}$$

Casimir Functionals

$$C_f = \int_{\mathcal{D}} f(\omega) \, \mathrm{d}\mathbf{r}$$

Energy balance in 2D

Of all the invariants, energy and enstrophy are the most important

Energy $E = \frac{1}{2}\int_{\mathcal{D}} |\mathbf{v}|^2 \, d\mathbf{r}$ Enstrophy $Z = \frac{1}{2}\int_{\mathcal{D}} \omega^2 \, d\mathbf{r}$

Energy and Enstrophy balance equations

$$\boxed{\frac{dE}{dt} = -2\nu Z \qquad \frac{dZ}{dt} = -2\nu P}$$

Palinstrophy

$$P = \frac{1}{2}\int |\nabla \times \boldsymbol{\omega}|^2 \, d\mathbf{r}$$

- As the palinstrophy is positive definite, the **total enstrophy cannot grow**

No energy dissipative anomaly in 2D

- As the enstrophy remains bounded, the **energy dissipation rate cannot remain finite in the inviscid limit**

$$\lim_{\nu \to 0} \frac{dE}{dt} = 0$$

2D turbulent cascades

Two quadratic invariants imply a double cascade Fjørtoft, Tellus, **5**, 225, (1953)

- Assume there exists **two sinks** and **one source** separated by two inertial ranges

Stationary state

$$\boxed{\epsilon_I = \epsilon_\alpha + \epsilon_\nu \qquad \eta_I = \eta_\alpha + \eta_\nu}$$

Characteristic scales

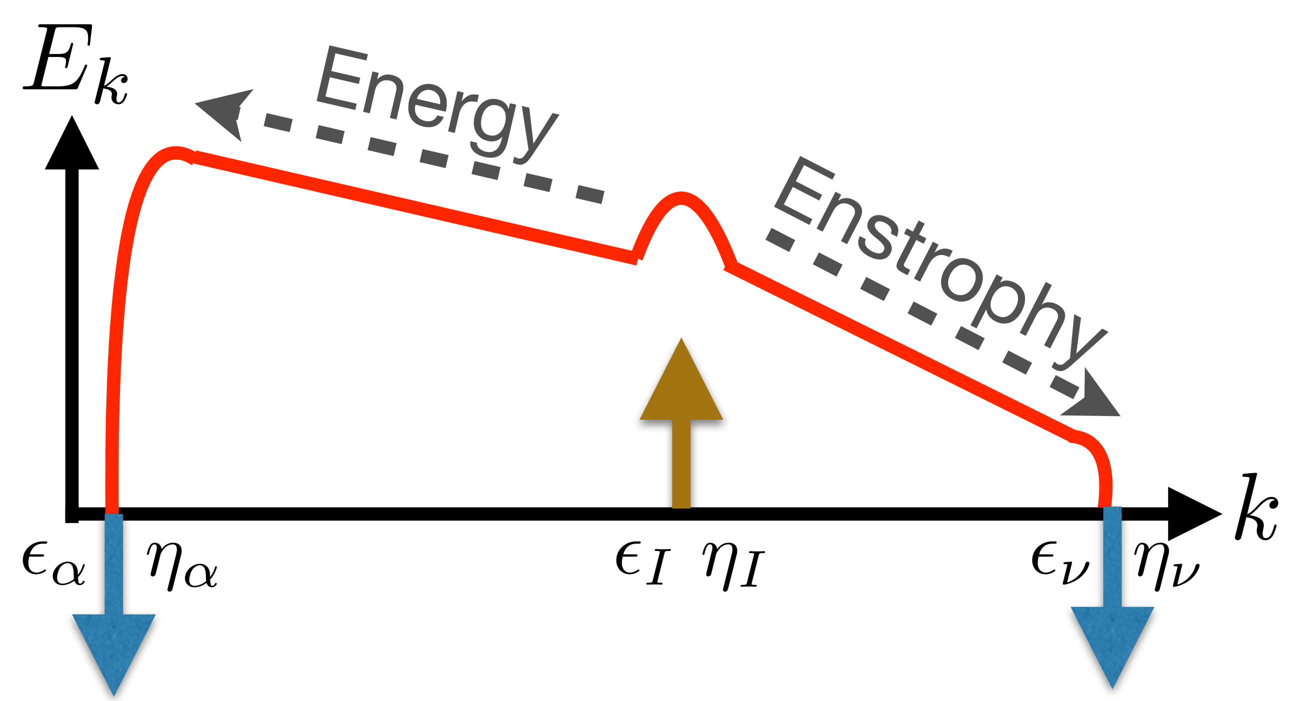
$$l_\alpha^2 = \epsilon_\alpha/\eta_\alpha \quad l_I^2 = \epsilon_I/\eta_I \quad l_\nu^2 = \epsilon_\nu/\eta_\nu$$

Enstrophy ratio

$$\frac{\eta_\alpha}{\eta_\nu} = \left(\frac{l_I}{l_\alpha}\right)^2 \frac{1-(l_\nu/l_I)^2}{1-(l_I/l_\alpha)^2} \xrightarrow{l_\nu \ll l_I \ll l_\alpha} 0$$

Energy ratio

$$\frac{\epsilon_\nu}{\epsilon_\alpha} = \left(\frac{l_\nu}{l_I}\right)^2 \frac{1-(l_I/l_\alpha)^2}{1-(l_\nu/l_I)^2} \xrightarrow{l_\nu \ll l_I \ll l_\alpha} 0$$



2D energy spectra

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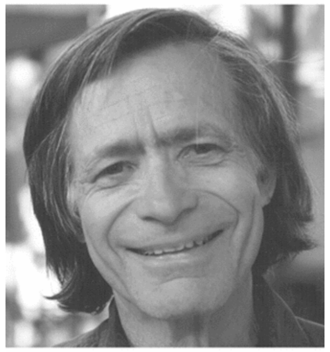
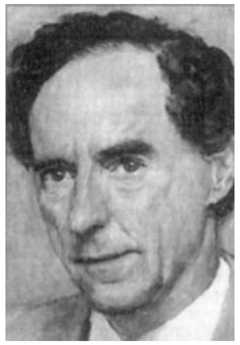
Kraichnan-Leith-Batchelor phenomenology (1967-1969)





- Following the results of Kolmogorov for 3D turbulence, it is possible to obtain equivalent results for 2D turbulence

Inverse energy cascade

$$\langle (\delta v_r)^3 \rangle = \frac{3}{2}\epsilon r$$

Energy spectrum scalings

$$\langle (\delta v_r)^2 \rangle \propto \epsilon^{2/3} r^{2/3} \quad \Rightarrow \quad E_k \propto \epsilon^{2/3} k^{-5/3}$$

Direct enstrophy cascade

$$\langle \delta v_r \, (\delta \omega_r)^2 \rangle = -2\eta r$$

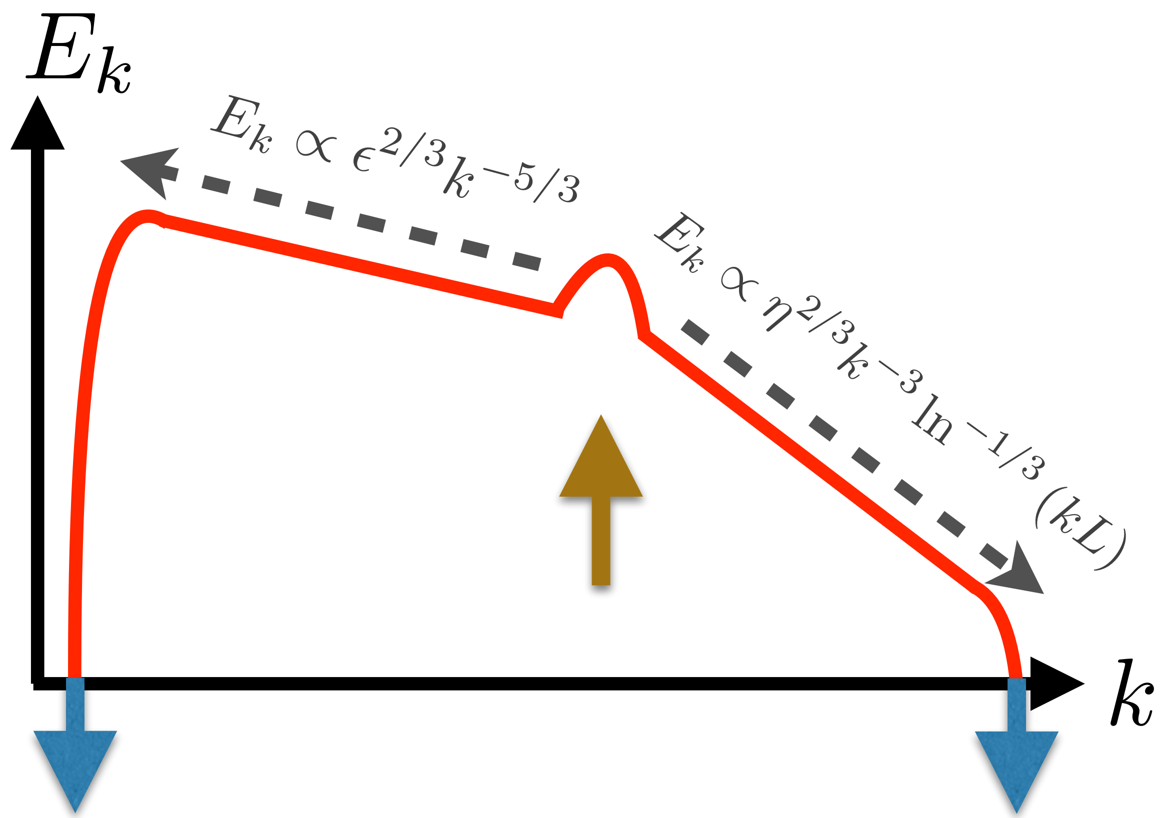
$$\langle (\delta v_r)^2 \rangle \propto \eta^{2/3} r^2 \quad \Rightarrow \quad E_k \propto \eta^{2/3} k^{-3}$$

Lindborg, J. Fluid Mech. **355**, 259-288, (1999)

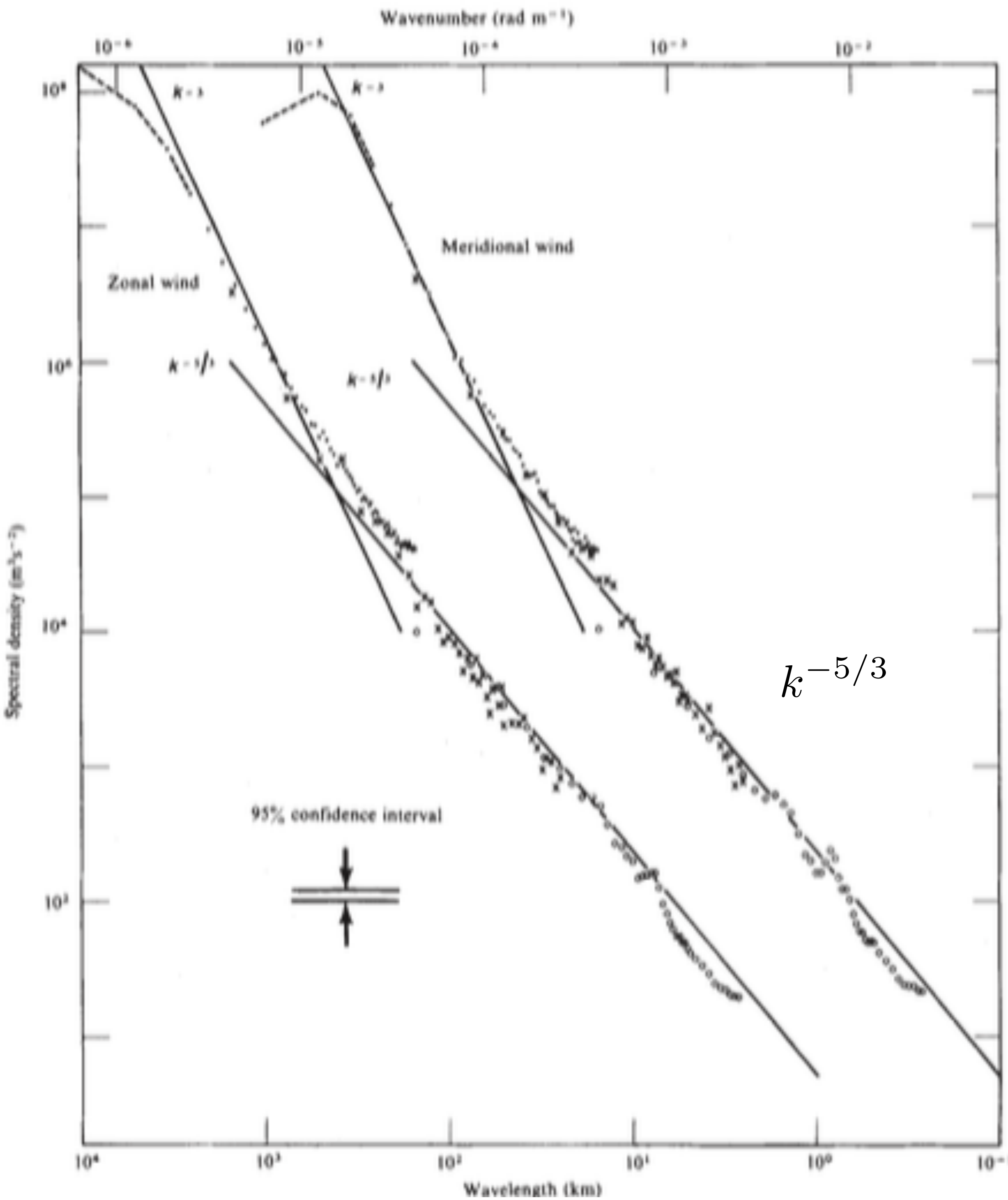
Kraichnan's logarithmic correction (1971)

- Total enstrophy **divergences** in large wavenumber limit: **nonlocality**
- Proposed a logarithmic correction to imply convergence

$$E_k \propto \eta^{2/3} k^{-3} \ln^{-1/3}(kL)$$



Evidence from atmospheric data



GASP aircraft data of wind speeds in tropopause

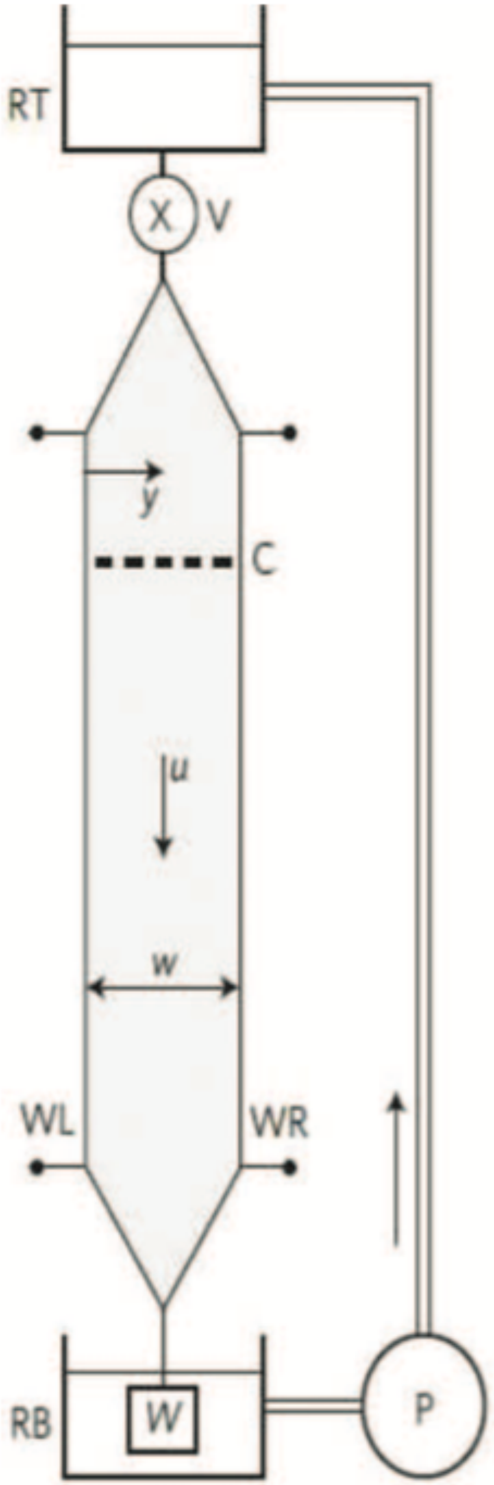
$k^{-5/3}$ observed for wavelength 3-300km

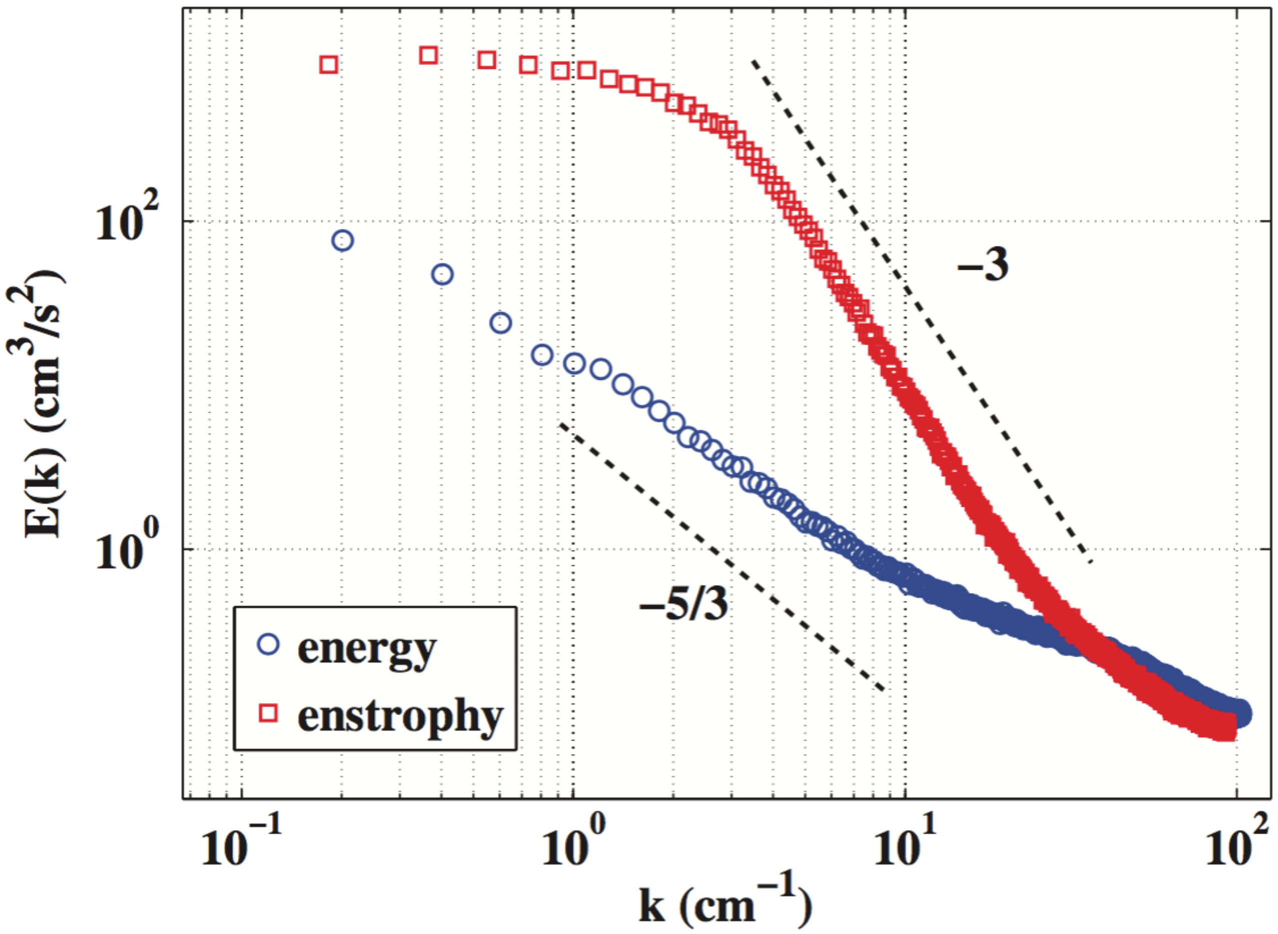
Nastrom *et al.* Nature, **312**, (1984)

Experimental evidence: soap films

Soap film experiments

Couder, Goldburg, Kellay, Rutgers, Rivera, Ecke,…





Cerbus and Goldburg. Phys. Fluids, **25**, (2013)

Experimental evidence: thin-layer fluids

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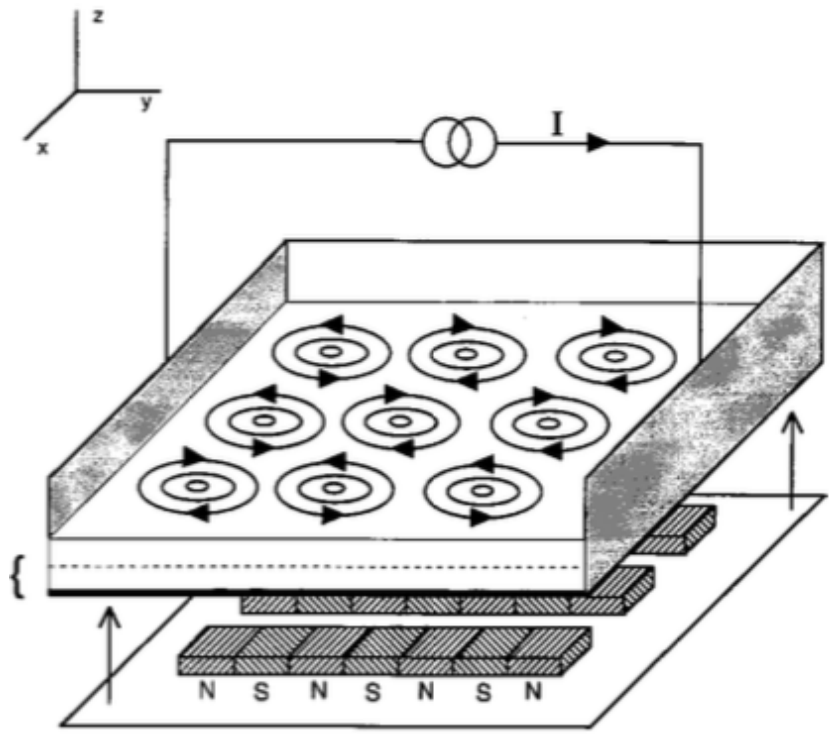
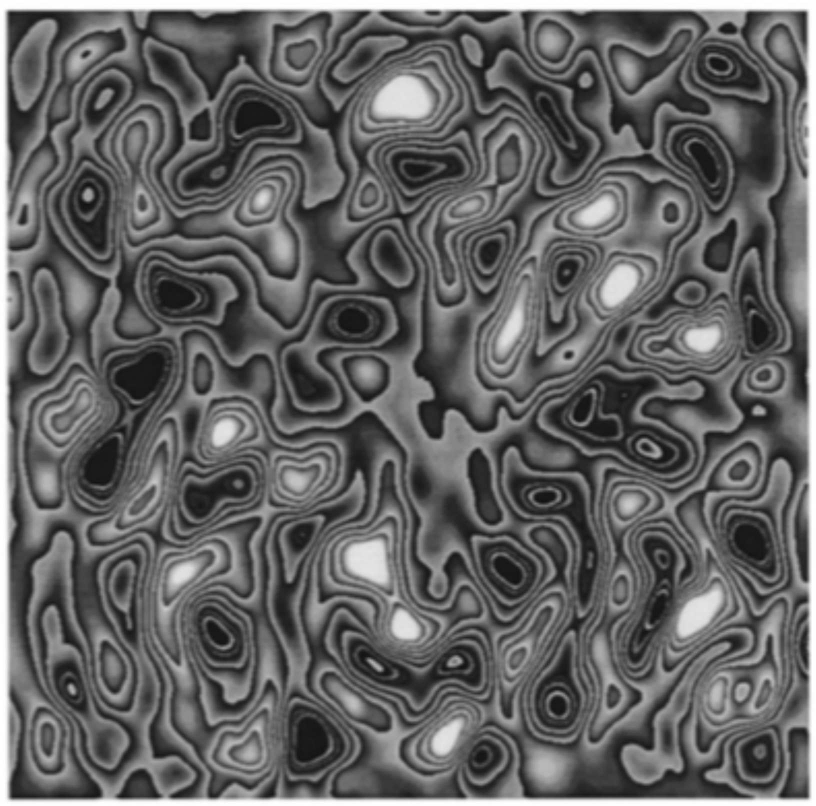
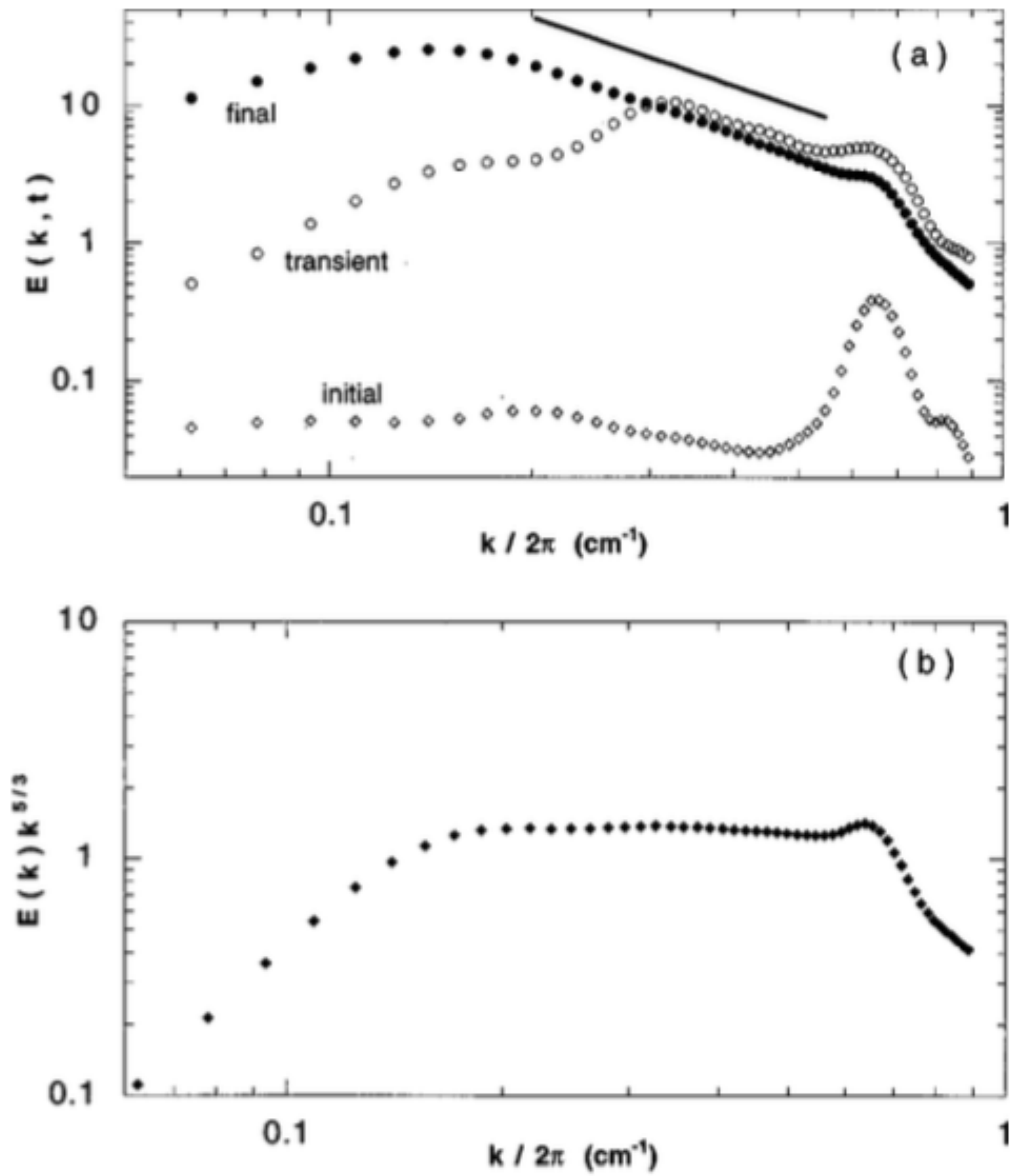
Thin layer electrolytes driven by a Lorenz force Sommeria, Tabeling, Gollub, Shats,…





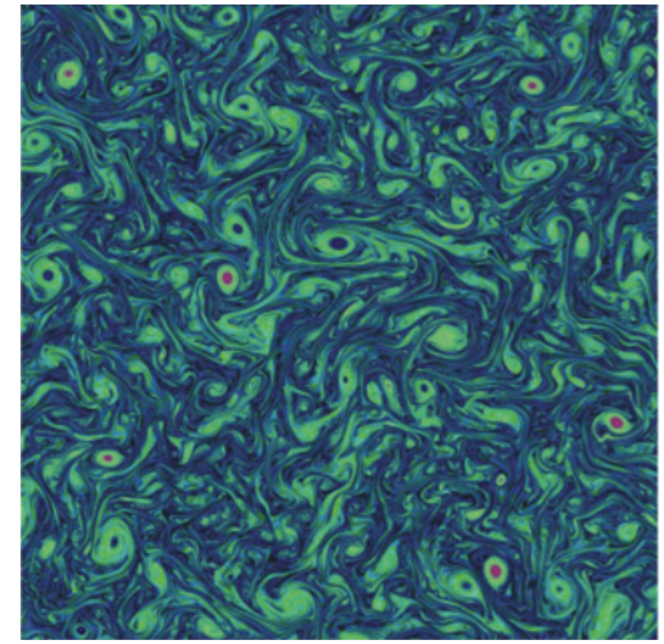


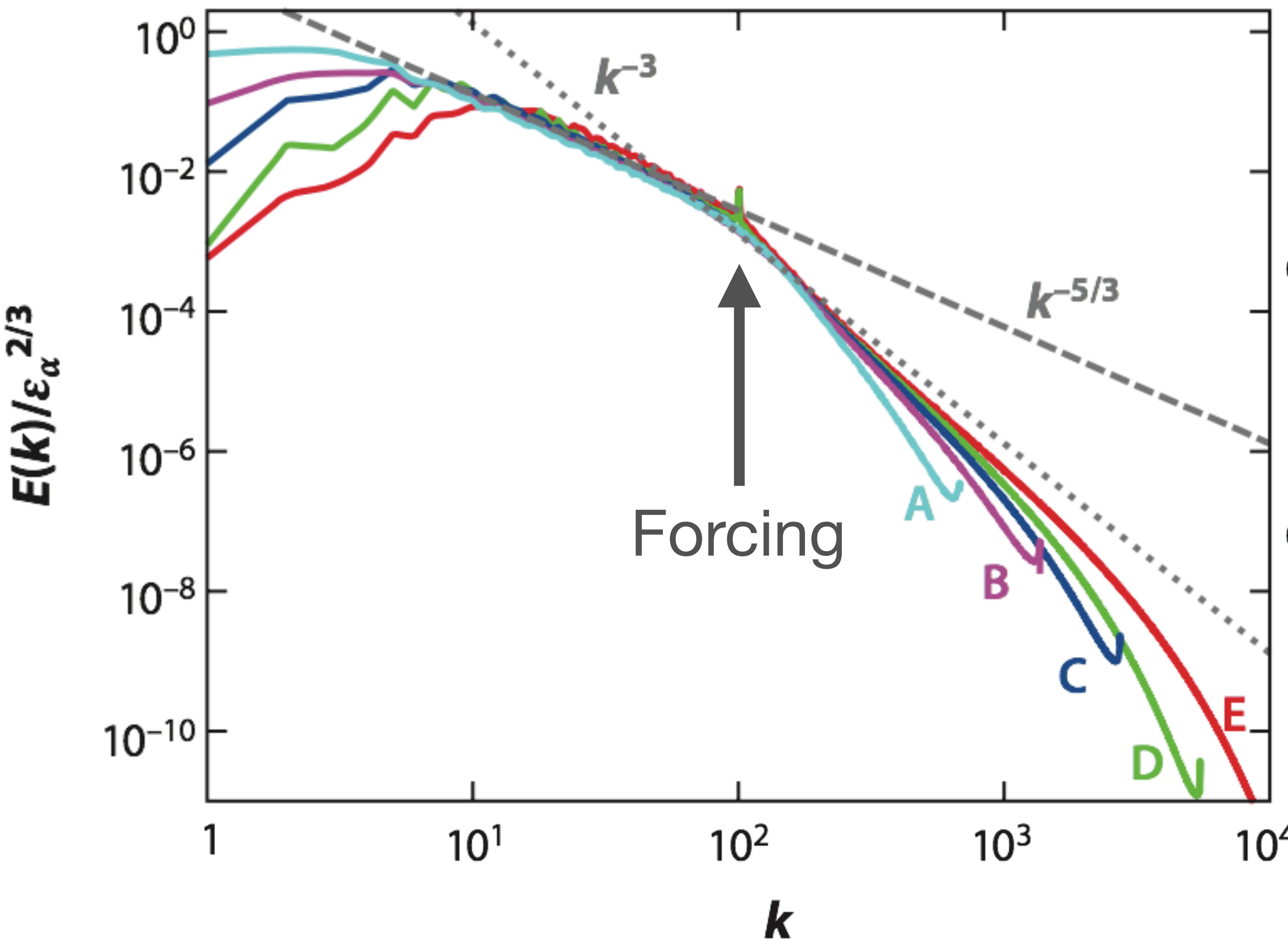
Paret and Tabeling. Phys. Rev. Lett. **79**, 4162, (1997)

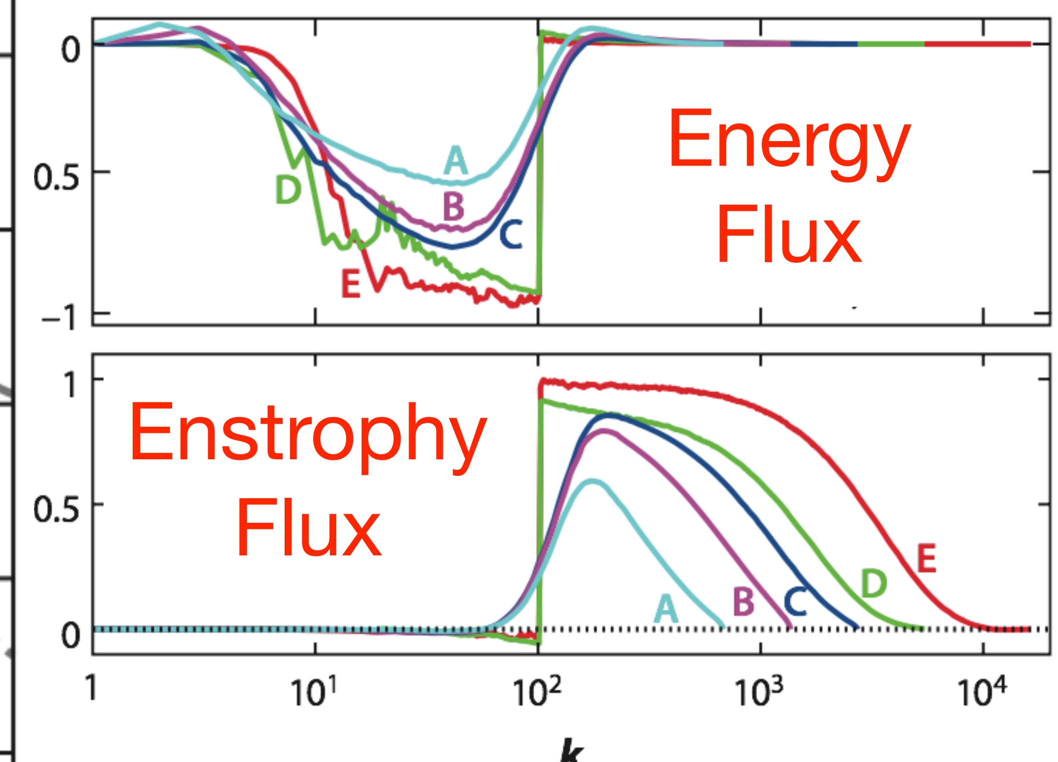
Numerical evidence

2D Navier-Stokes with linear friction

$$\frac{\partial \omega}{\partial t} + \mathbf{v} \cdot \nabla \omega = \nu \nabla^2 \omega - \alpha \omega + f_\omega$$







Boffetta and Musacchio,
Phys. Rev. E, **82**, 016307, (2010)

2D structure functions

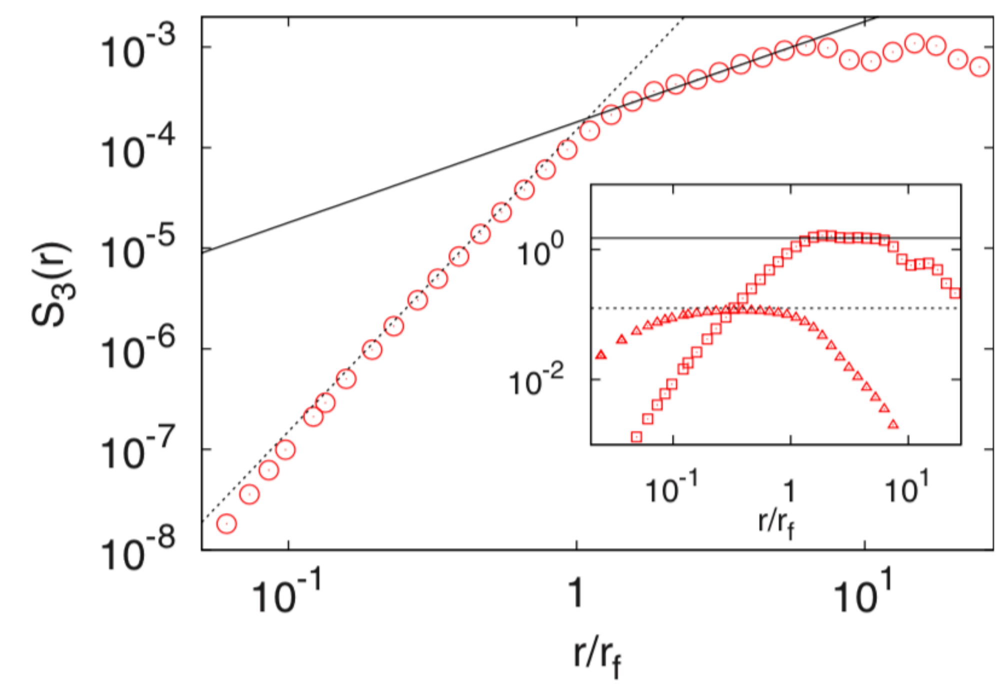
Third order structure function

Boffetta and Musacchio, Phys. Rev. E, **82**, 016307, (2010)

$$\left\langle (\delta v_r)^3 \right\rangle = \begin{cases} \frac{1}{8}\eta r^3 & \text{for } r \ll l_f \\ \frac{3}{2}\epsilon r & \text{for } l_f \ll r \end{cases}$$

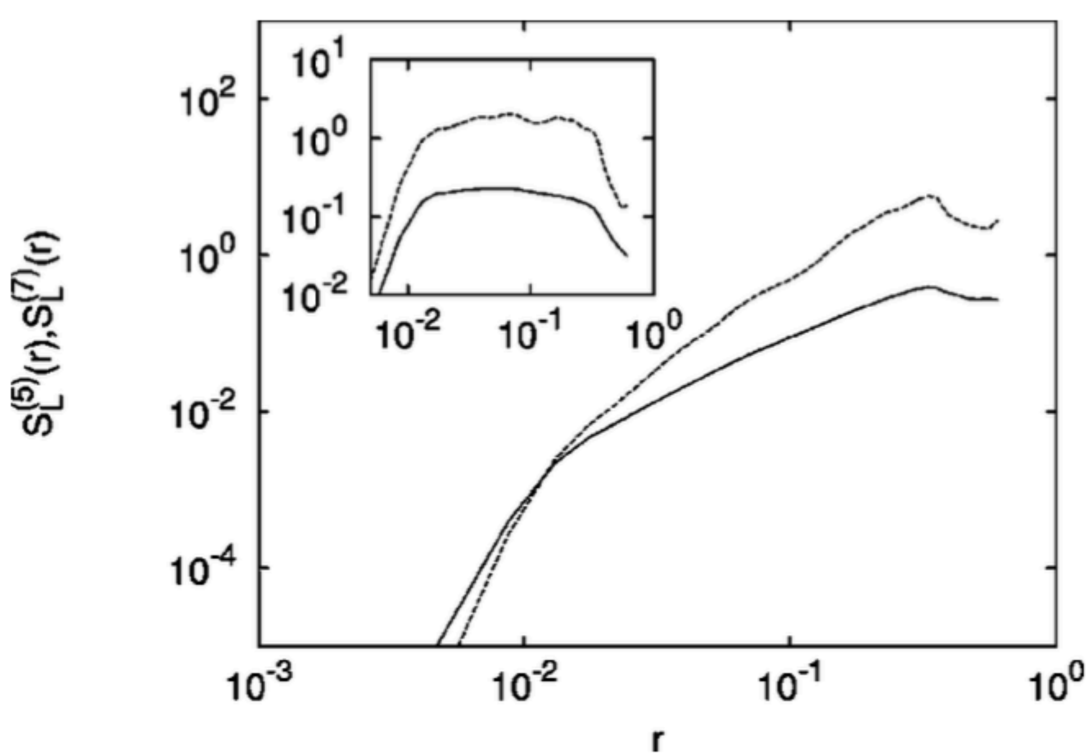
Direct enstrophy cascade

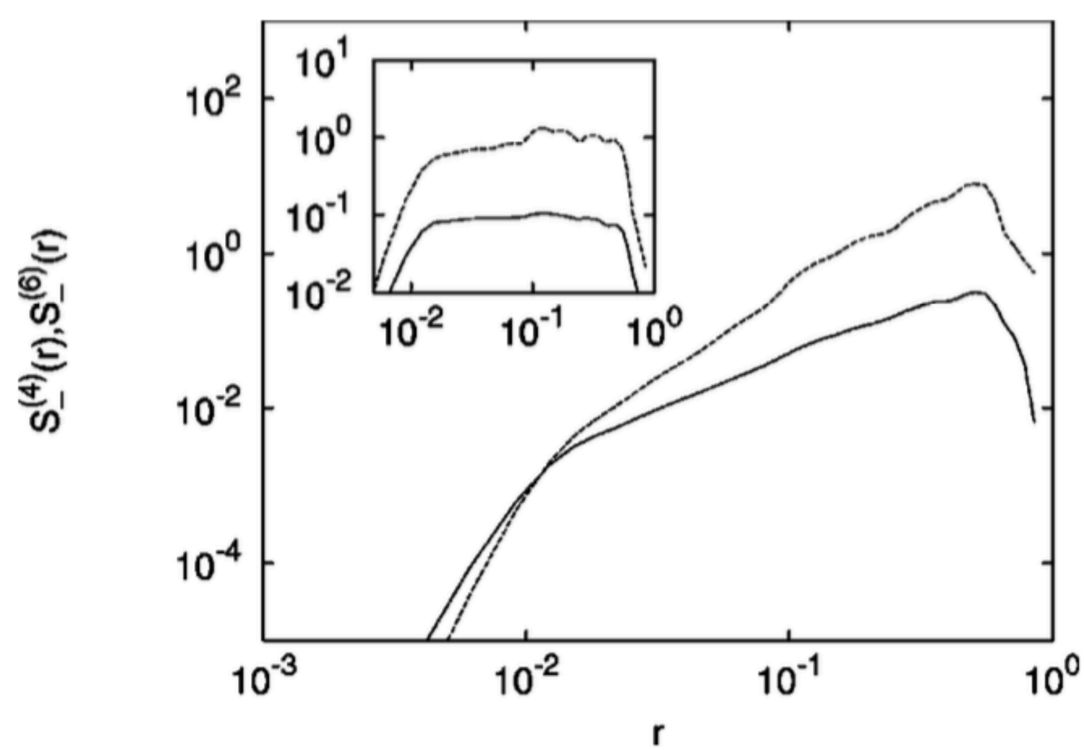
Inverse energy cascade



Higher order structure functions

Boffetta, Celani and Vergassola, Phys. Rev. E, **61**, 29, (2000)





Structure function scalings compatible with $\langle (\delta v_r)^p \rangle \propto (\epsilon r)^{p/3}$ - no intermittency!

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2D Energy Condensation and
Large-Scale Mean Flows

Energy condensation in 2D

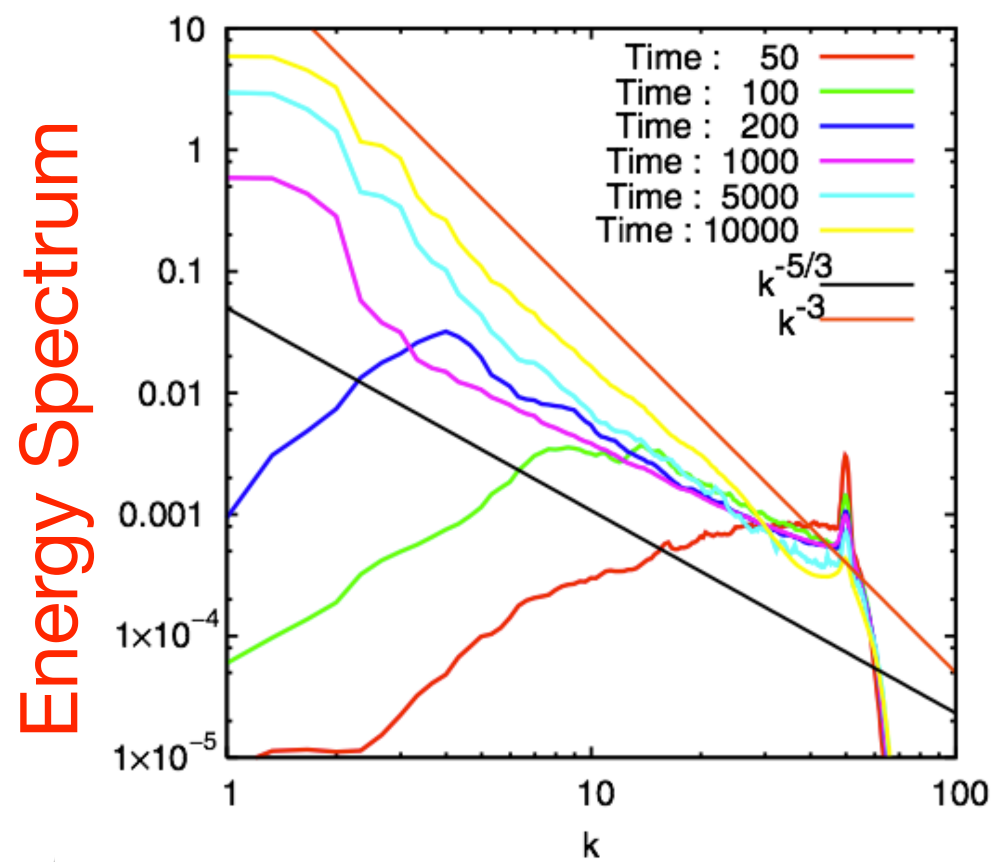
Realizability of the inverse cascade of energy

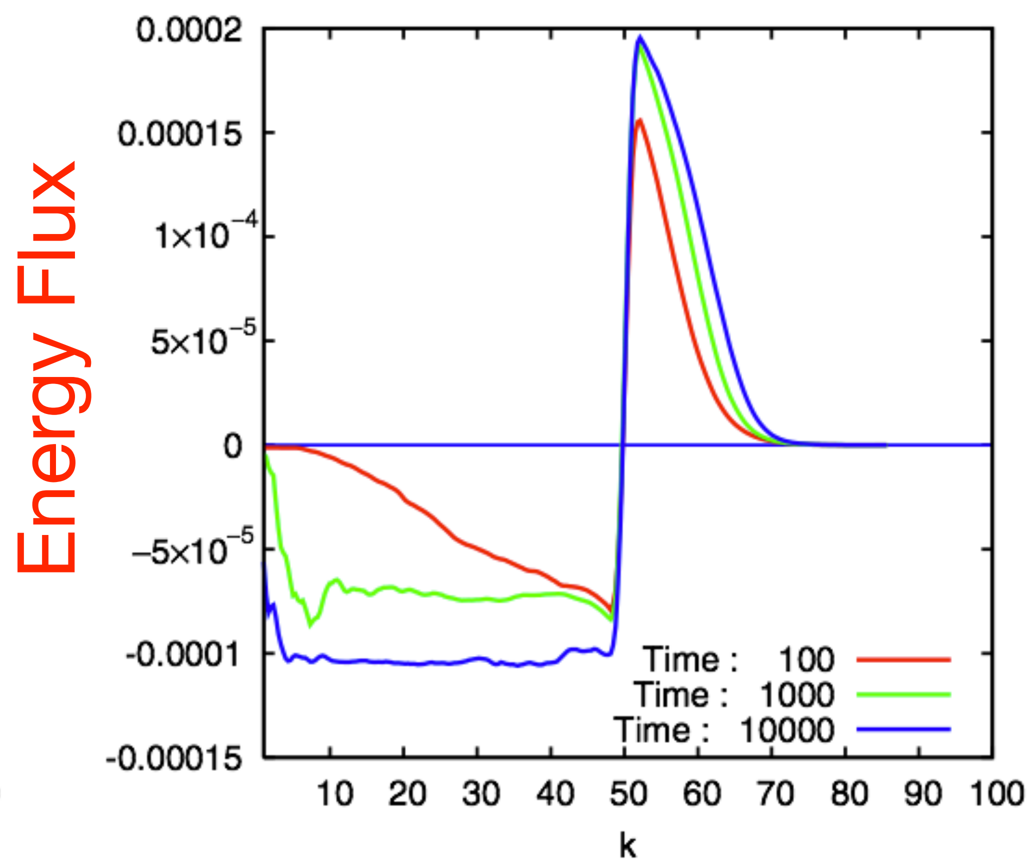
- Infinite sized systems
- Finite size with sufficient large-scale dissipation

Otherwise…spectral condensation

- Inverse cascade reaches the largest scale and is **blocked**
- Energy will continuously be fed into the largest modes
- Observed $E_k \propto k^{-3}$ behaviour

Forced 2D Navier-Stokes without linear friction





Chertkov *et al.* Phys. Rev. Lett. **99**, 084501, (2007)

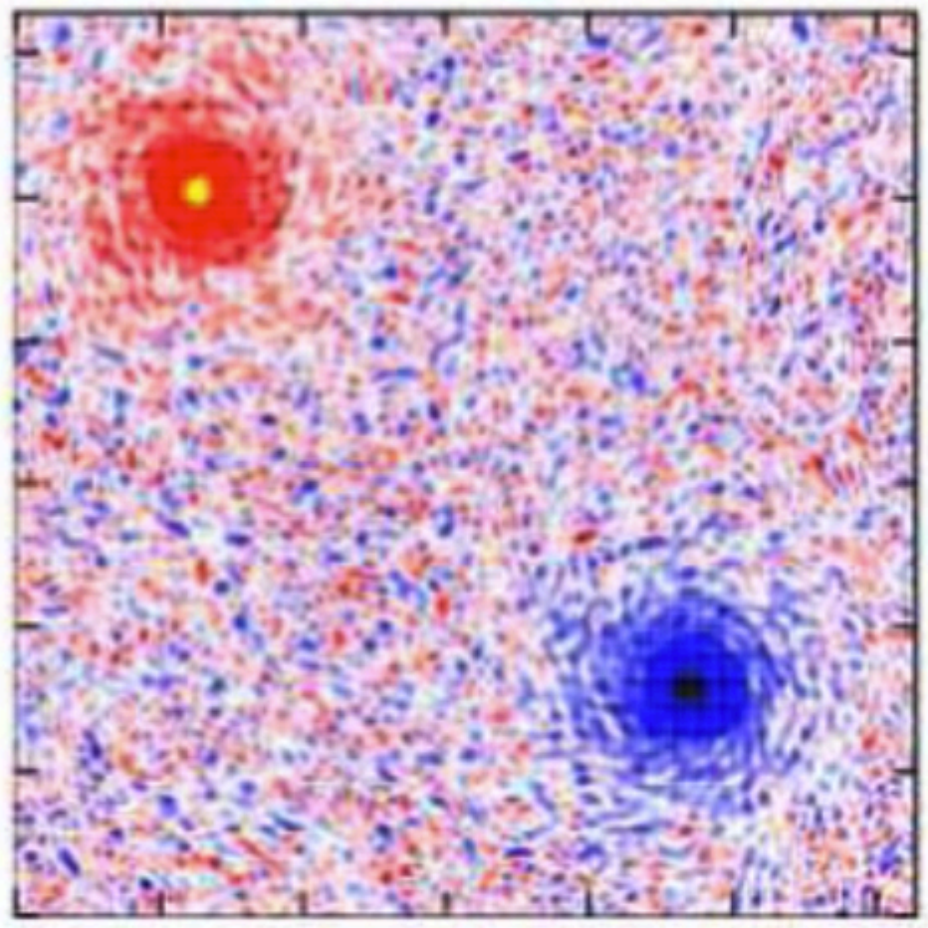
Energy condensation in 2D

Spectral condensation leads to spatial self-organization of the flow

- Form of mean flow is dependent on domain and boundary conditions

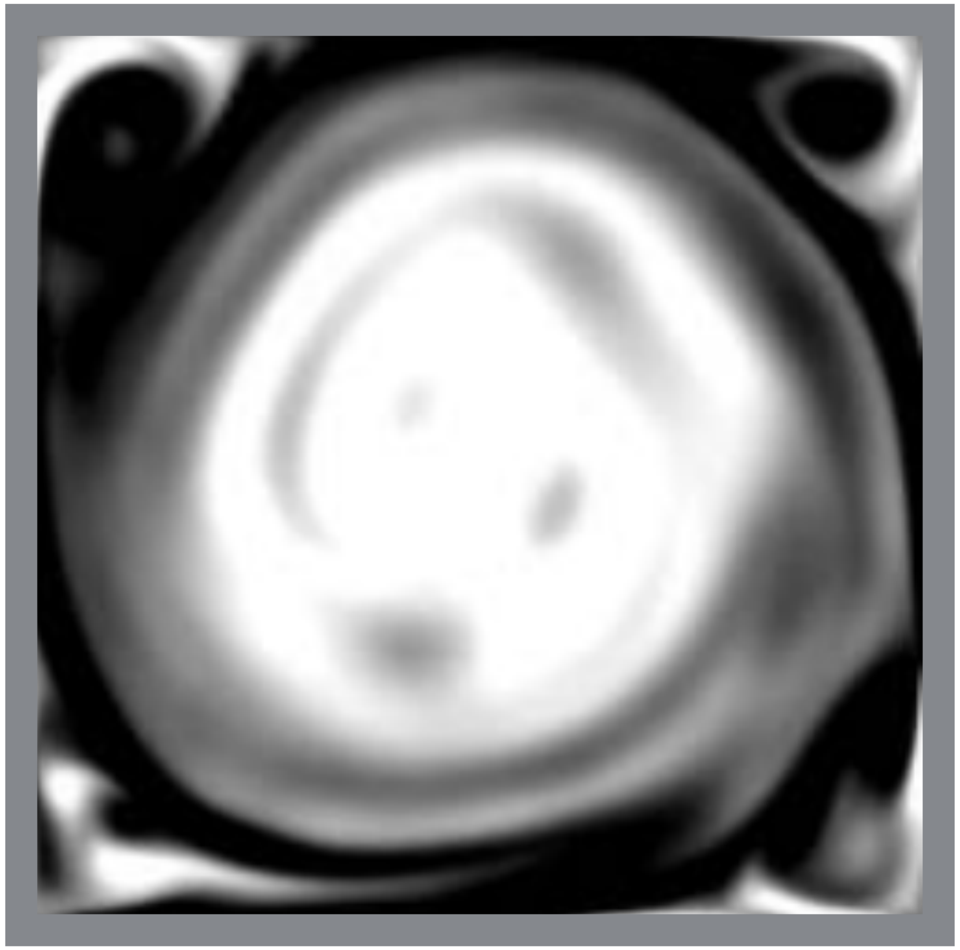
2D Navier-Stokes simulations

Periodic boundaries



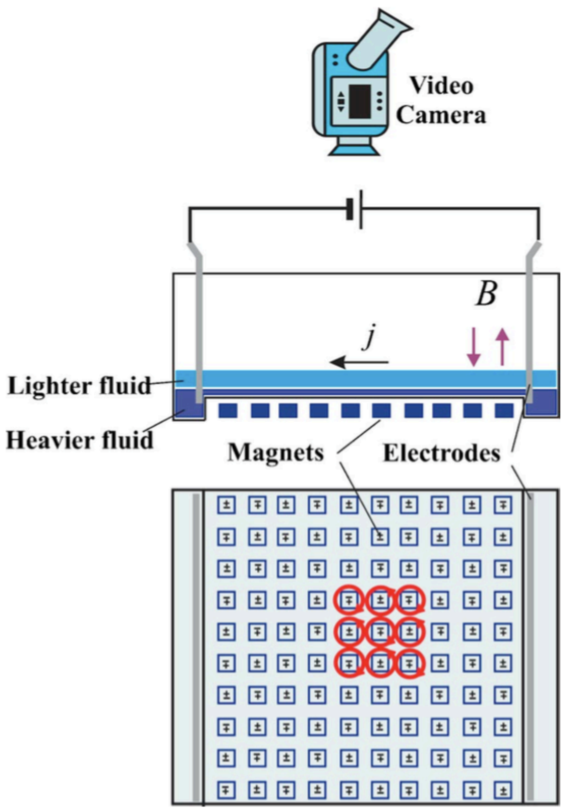
Chertkov *et al.* Phys. Rev. Lett. **99**, 084501, (2007)

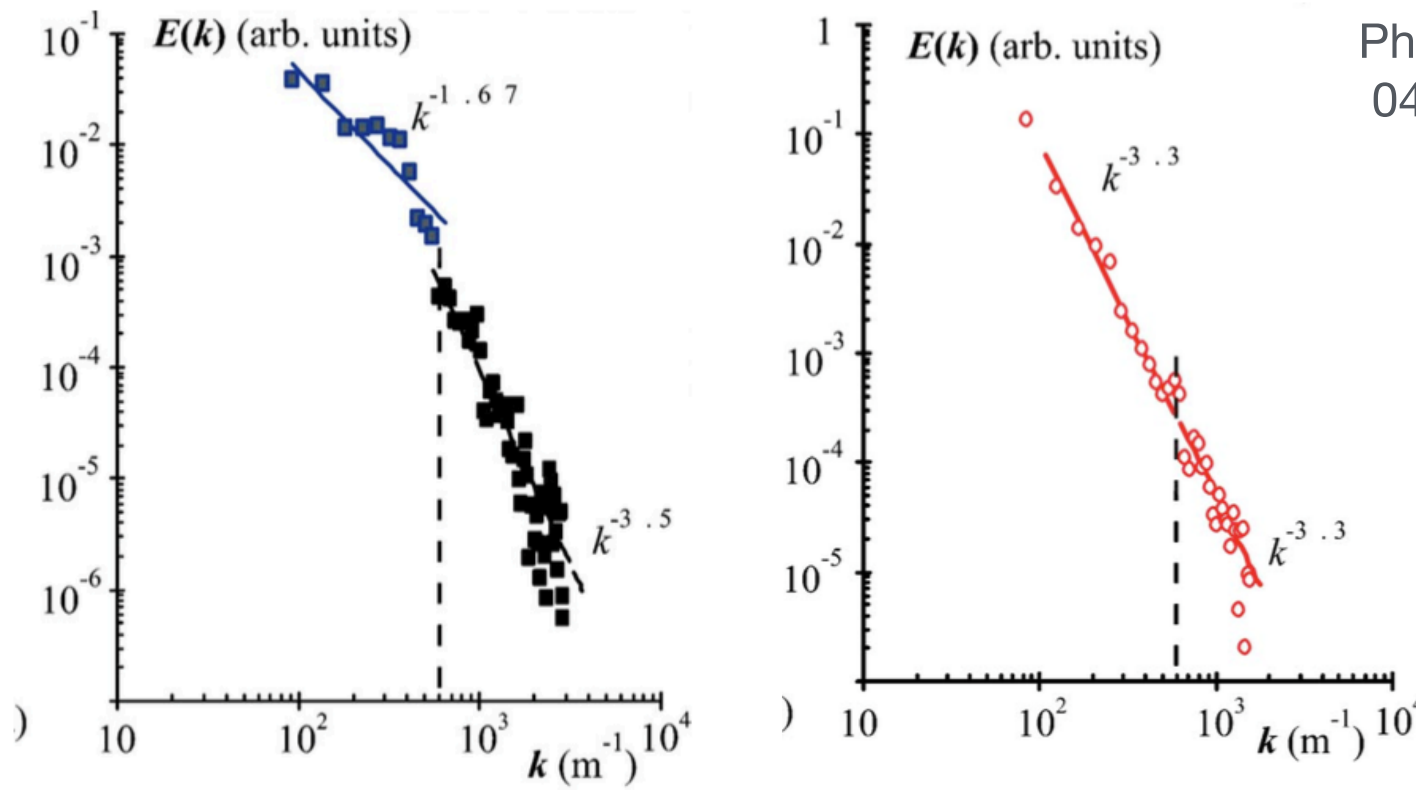
No-slip boundaries



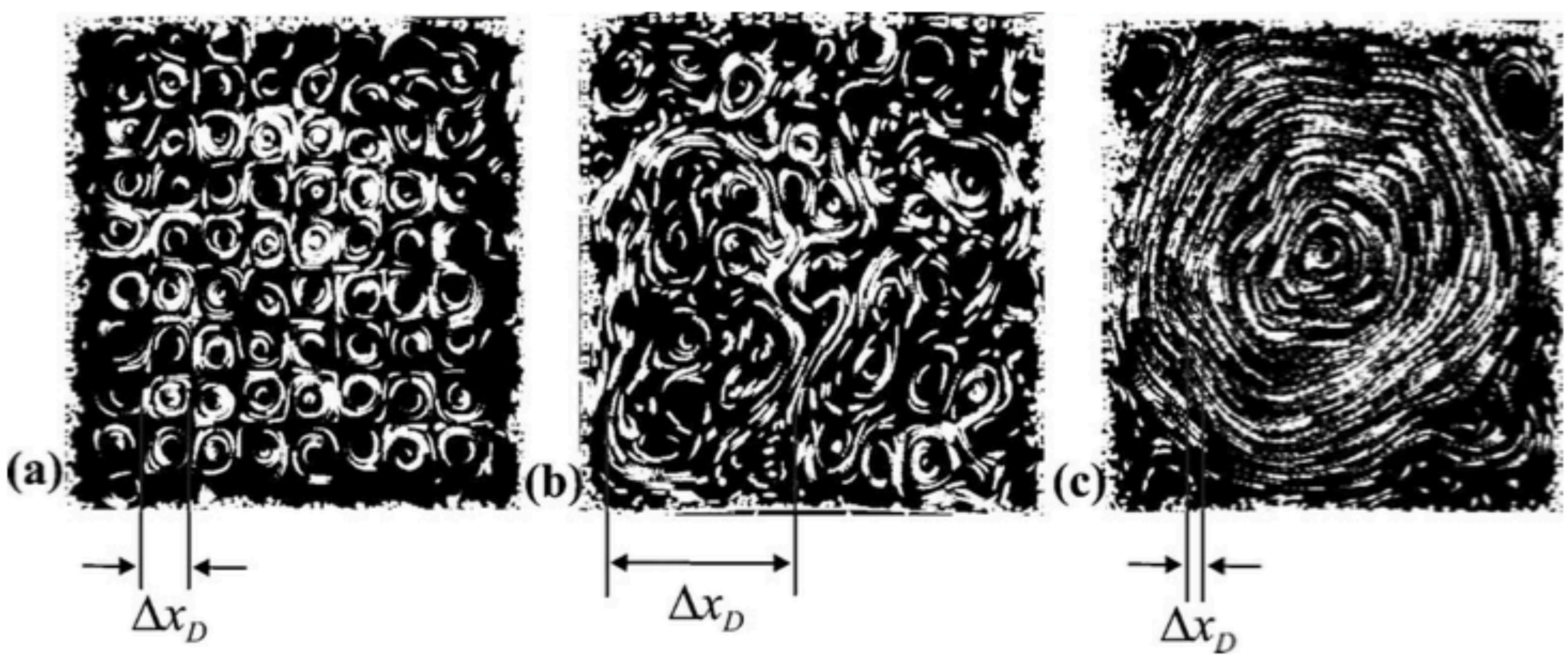
van Heijst *et al.* J. Fluid Mech. **554**, 411, (2006)

Energy condensation in experiments





Shats *et al.*
Phys. Rev. E, **71**, 046409, (2005)



How to predict the condensate?

Equilibrium statistical mechanics: Miller-Robert-Sommeria (1990, 1991)

- Argument justified for dynamical systems relaxing toward equilibrium (Euler dynamics)

- $\rho(\mathbf{r}, \Omega)$ is the local probability to have $\omega(\mathbf{r}) = \Omega$ at position $\mathbf{r}$

Miller, Phys. Rev. Lett. **65**, 2137, (1990)
Robert, Sommeria, J. Fluid Mech. **229**, 291, (1991)

Microcanonical variational problem

$$S(E, \gamma) = \sup_{\rho} \left\{ \int \int_{-\infty}^{\infty} \rho \ln \rho \, \mathrm{d}\Omega \, \mathrm{d}\mathbf{r} \quad \Big| \quad \mathcal{E}[\rho] = E, \; D[\rho] = \gamma \right\}$$

- In principle this extremely tough problem
- However, it can be shown that **entropy maximisers** satisfy

$$\omega = f(\beta\psi) \quad \Rightarrow \quad \mathbf{v} \cdot \nabla\omega = 0$$

Energy-Casimir variational problem

$$C(E, s) = \inf_{\omega} \left\{ \mathcal{C}_s[\omega] = \int_{\mathcal{D}} s(\omega) \, \mathrm{d}\mathbf{r} \quad \Big| \quad \mathcal{E}[\omega] = E \right\}$$

- **Solutions of the EC-VP are solutions of the MVP** for the particular energy and casimirs

How to predict the condensate?

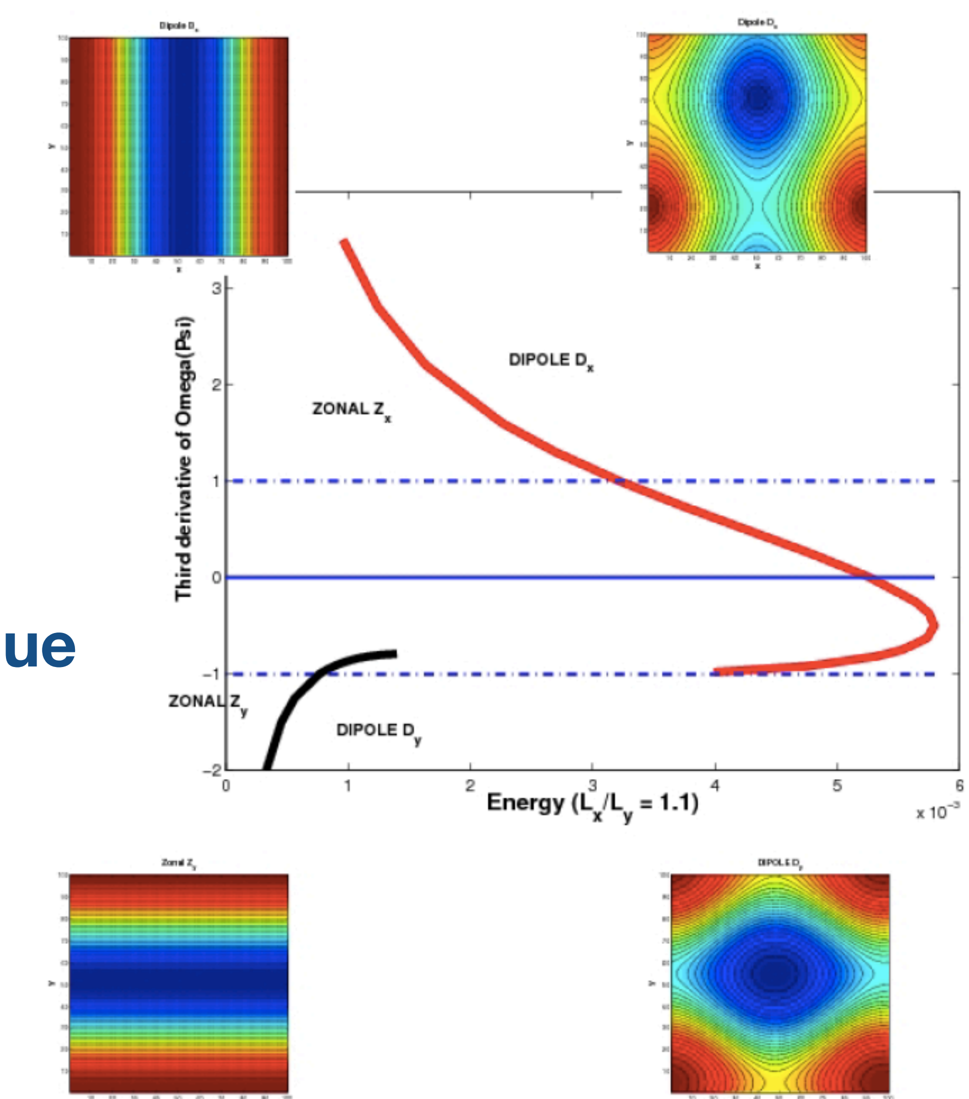
Energy-Casimir variational problem

$$C(E,s) = \inf_{\omega} \left\{ \mathcal{C}_s[\omega] = \int_{\mathcal{D}} s(\omega)\,\mathrm{d}\mathbf{r} \quad \middle| \quad \mathcal{E}[\omega] = E \right\}$$

Solution in the weak energy limit

Bouchet and Venaille, Phys. Rep. **515**, 227 (2012)

$$s(\omega) = \frac{\omega^2}{2} + \frac{a_4 \omega^4}{4} + o(\omega^5)$$



- At leading order, enstrophy becomes the most important casimir and we get a linear relationship

$$\omega = \beta\psi$$

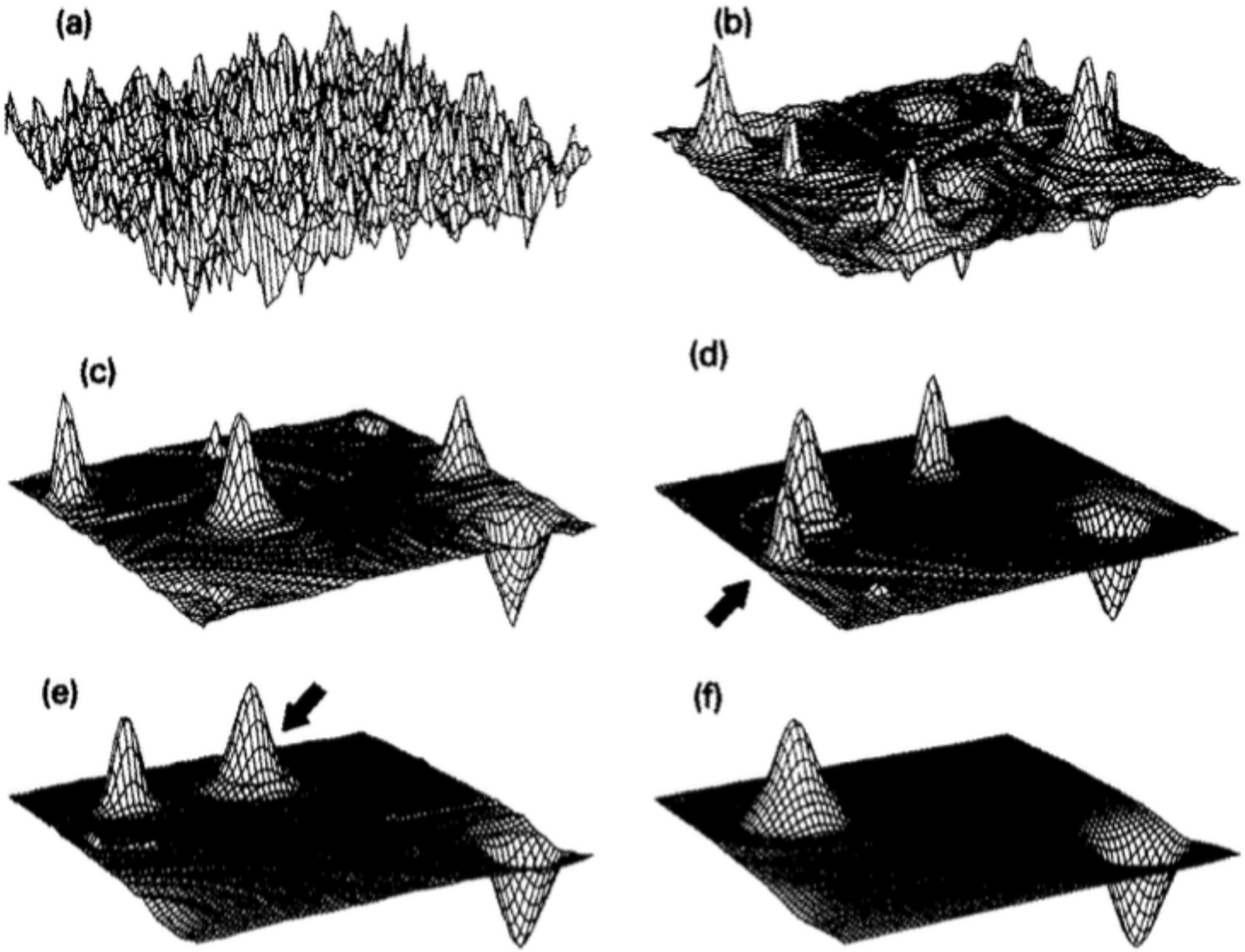
- Leads to **largest-scale argument** where energy is situated at the **eigenmodes with smallest eigenvalue**

$$\omega = A\cos(x) + B\cos(y)$$

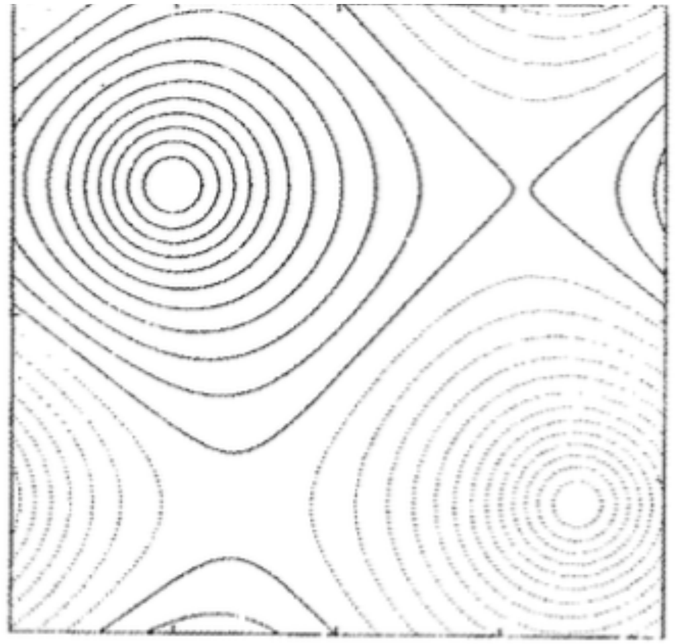
- Degeneracy is removed by considering next order casimir or aspect ratios > 1

Decaying 2D Navier-Stokes turbulence

Dipole appears at largest scale as flow decays

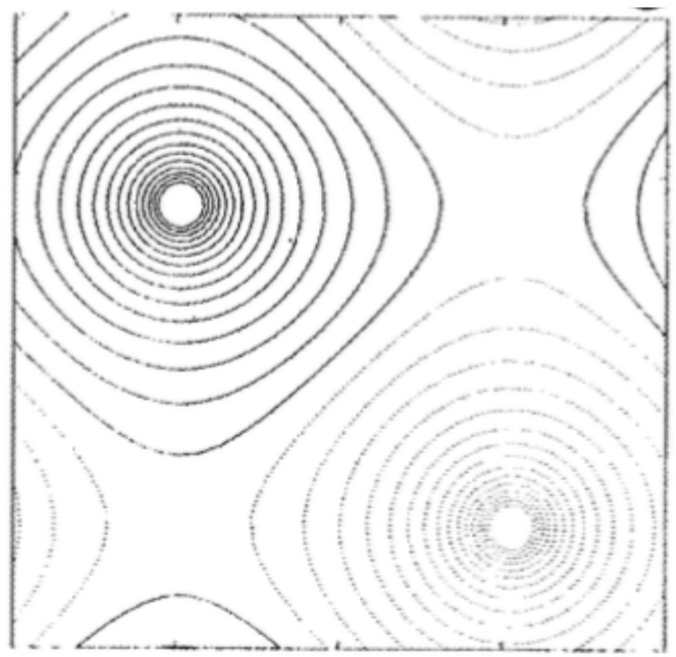


Numerics



Theory

$$\omega = A\cos(x) + B\cos(y)$$

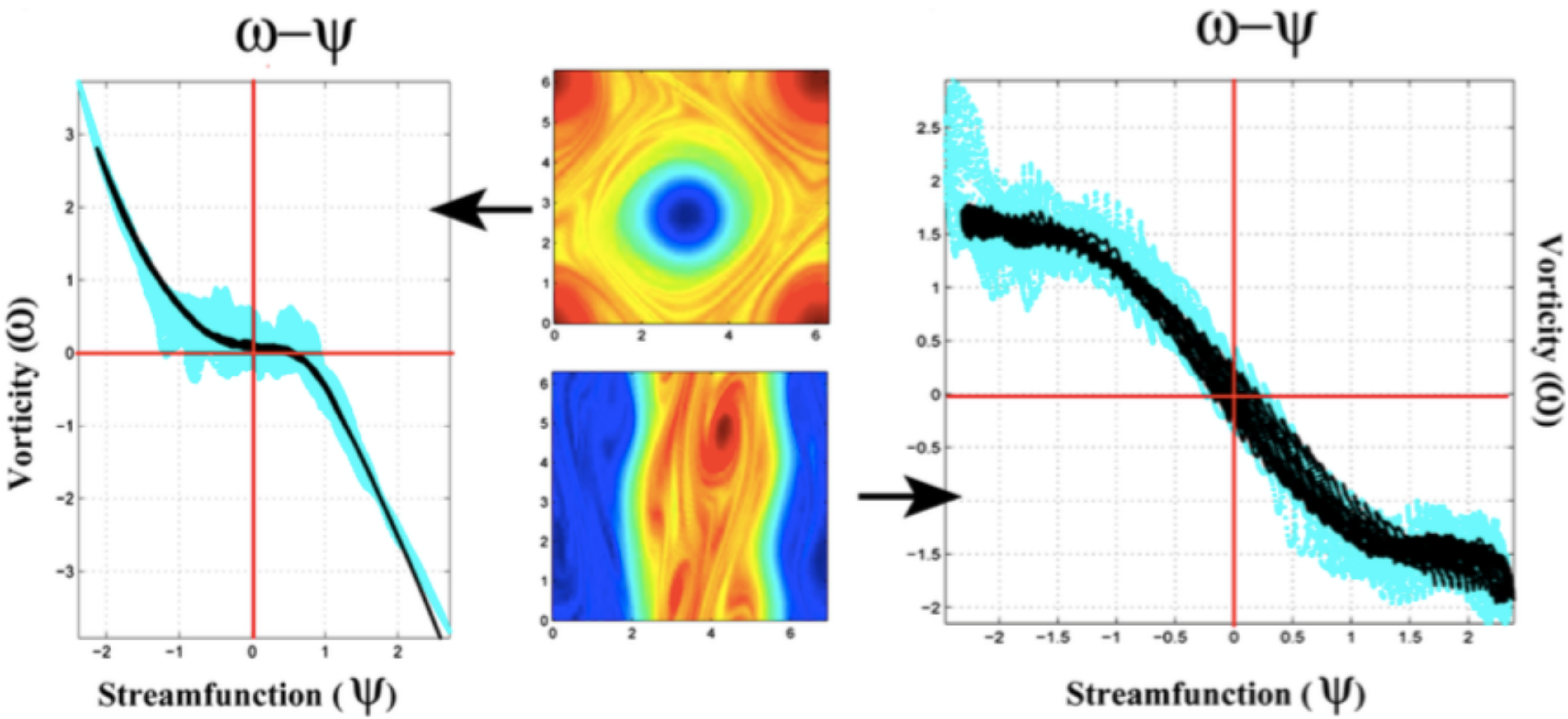


Matthaeus *et al.* Physica D, **51**, 531, (1991)

Forced 2D Navier-Stokes turbulence

Stochastically forced 2D Navier-Stokes with linear friction

Bouchet and Simonnet, Phys. Rev. Lett. **102**, 094504, (2009)



- Both dipole and zonal jets appear depending on the aspect ratio of periodic domain
- **Vorticity-Streamfunction relation** is **nonlinear**:

Mean flow not solely contained in largest modes

Vortex profile in forced 2D turbulence

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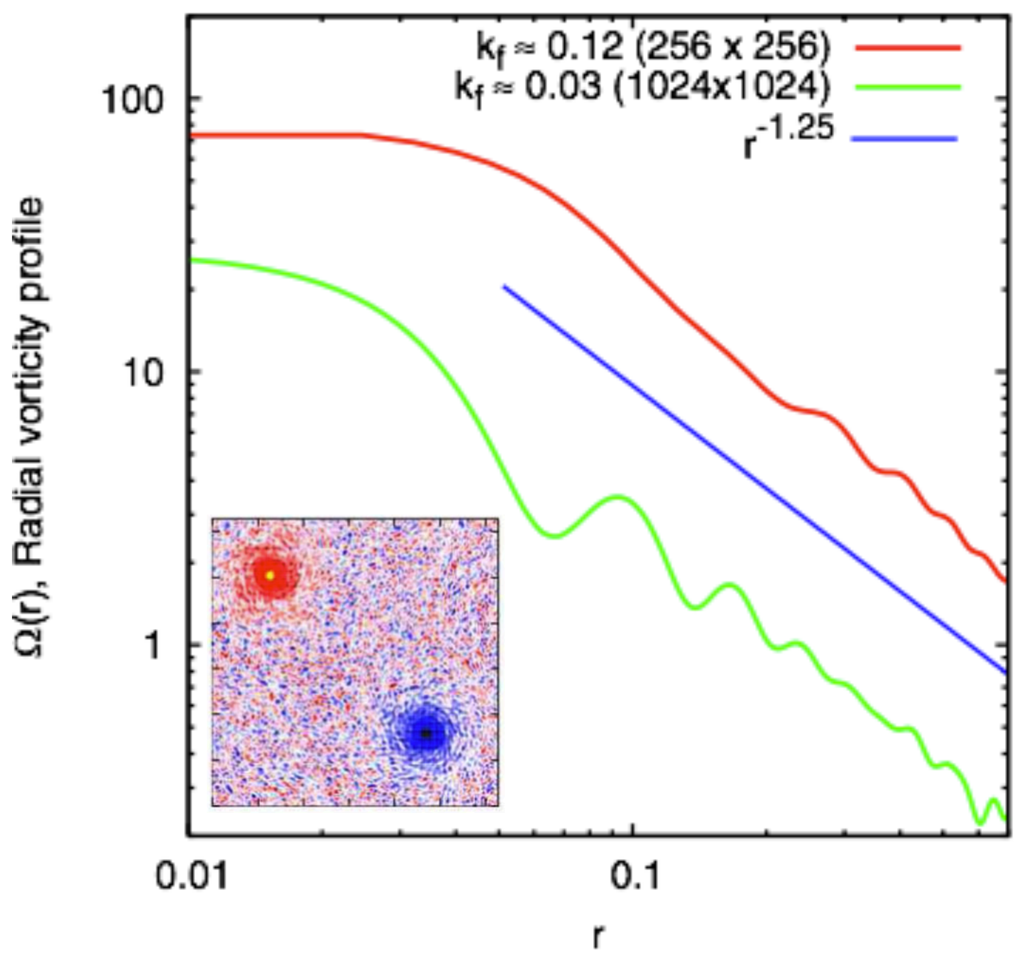
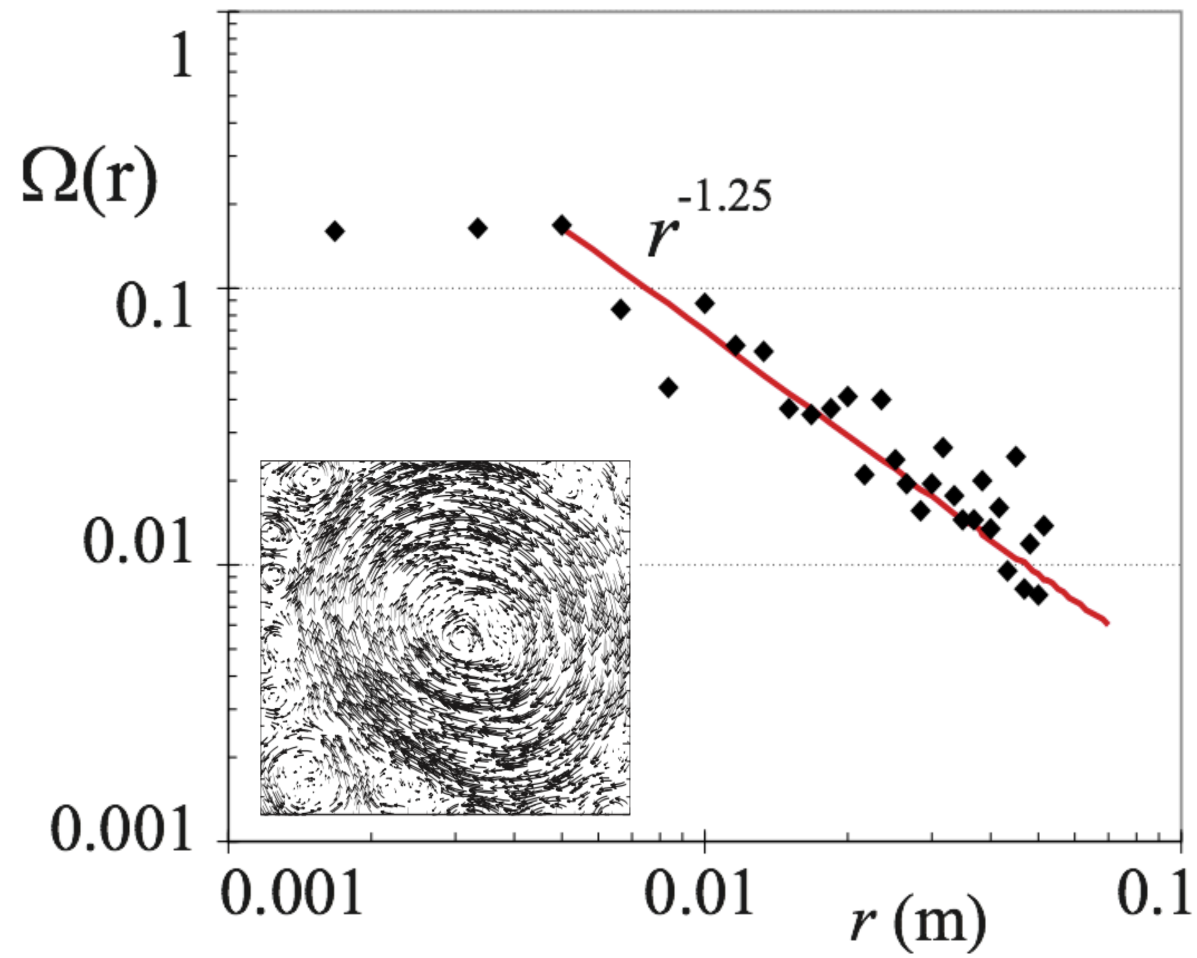
2D Navier-Stokes equations

Chertkov *et al.* Phys. Rev. Lett. **99**, 084501, (2007)



Thin-layer experiment

Xia *et al.* Phys. Fluids, **21**, 125101, (2009)



Observations

- Mean vorticity scaling appears to be $\Omega \propto r^{-5/4}$
- Largest scale argument is **insufficient** to predict profile: does not lead to $\Omega \propto r^{-5/4}$
- A **non-equilibrium approach is likely needed** to explain mean flow structure

Mean flow energy/momentum balance

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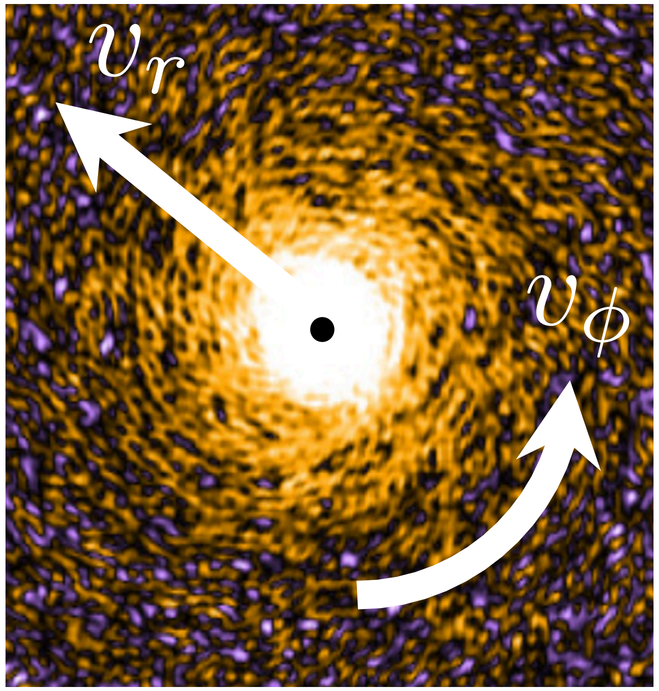
Reynolds flow decomposition



- Decompose flow into its temporal mean and fluctuating components using polar coordinates

$$\mathbf{v} = (v_\phi, v_r) = (U(r) + u(\phi, r, t), v(\phi, r, t))$$

$$\langle \mathbf{v} \rangle = (U(r), 0) \qquad \langle u \rangle = \langle v \rangle = 0$$

Momentum balance

$$\partial_r \langle r v^2 \rangle + r \partial_r \langle p \rangle = U^2 + \langle u^2 \rangle$$ radial component

$$\frac{1}{r} \partial_r \left(r^2 \langle uv \rangle \right) = -\alpha r U$$ azimuthal component

Energy balance

$$\frac{1}{r} \partial_r \left(r U \langle uv \rangle \right) = r \langle uv \rangle \partial_r \left(\frac{U}{r} \right) - \alpha U^2$$ Energy balance of mean flow

$$\frac{1}{r} \partial_r \left[r \left\langle v \left(\frac{u^2 + v^2}{2} + p \right) \right\rangle \right] = \epsilon - \alpha \langle u^2 + v^2 \rangle - r \langle uv \rangle \partial_r \left(\frac{U}{r} \right)$$ Energy balance of fluctuations

Vortex profile prediction

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Neglect higher-order turbulent velocity correlators

- We have a natural small parameter $\alpha^3 L^2/\epsilon \ll 1$ that relates the strength of the mean flow shear to that of the turbulence fluctuations
- Further assume that $\langle vp \rangle$ can also be **neglected inside vortex**

Power-law solutions of energy and momentum balance equations

$$\epsilon = \frac{1}{r}\partial_r \left(rU\langle uv\rangle\right) + \alpha U^2$$

Energy balance

$$\frac{1}{r}\partial_r \left(r^2\langle uv\rangle\right) = -\alpha r U$$

Momentum balance (azimuthal component)

JL *et al.* Phys. Rev. Lett. **113**, 254503, (2014)

$$\boxed{\begin{array}{ll} U = \sqrt{3\epsilon/\alpha} & \Omega = \sqrt{3\epsilon/\alpha}\; r^{-1} \\ \langle uv\rangle = \sqrt{\frac{\alpha\epsilon}{3}r} & \frac{1}{r}\partial_r\left(rU\langle uv\rangle\right) = -2\epsilon \end{array}}$$

- Not only scaling, but **also numerical prefactors are predicted!**
- Notice the shallower mean vorticity scaling to what was previously observed

Mean vortex velocity data

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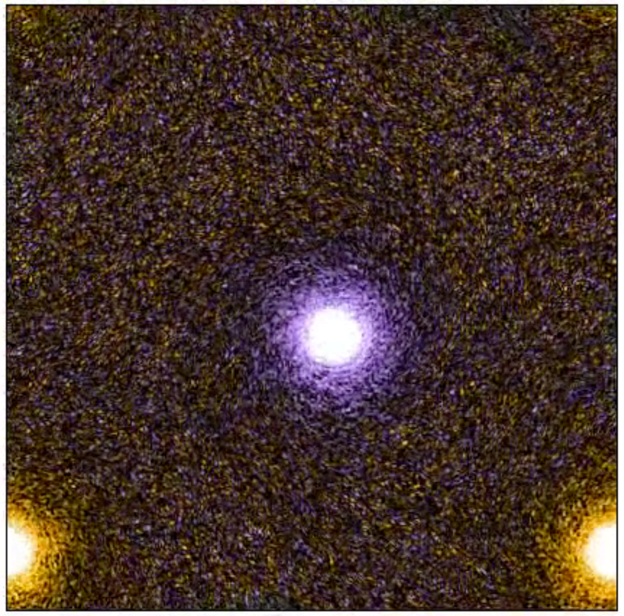
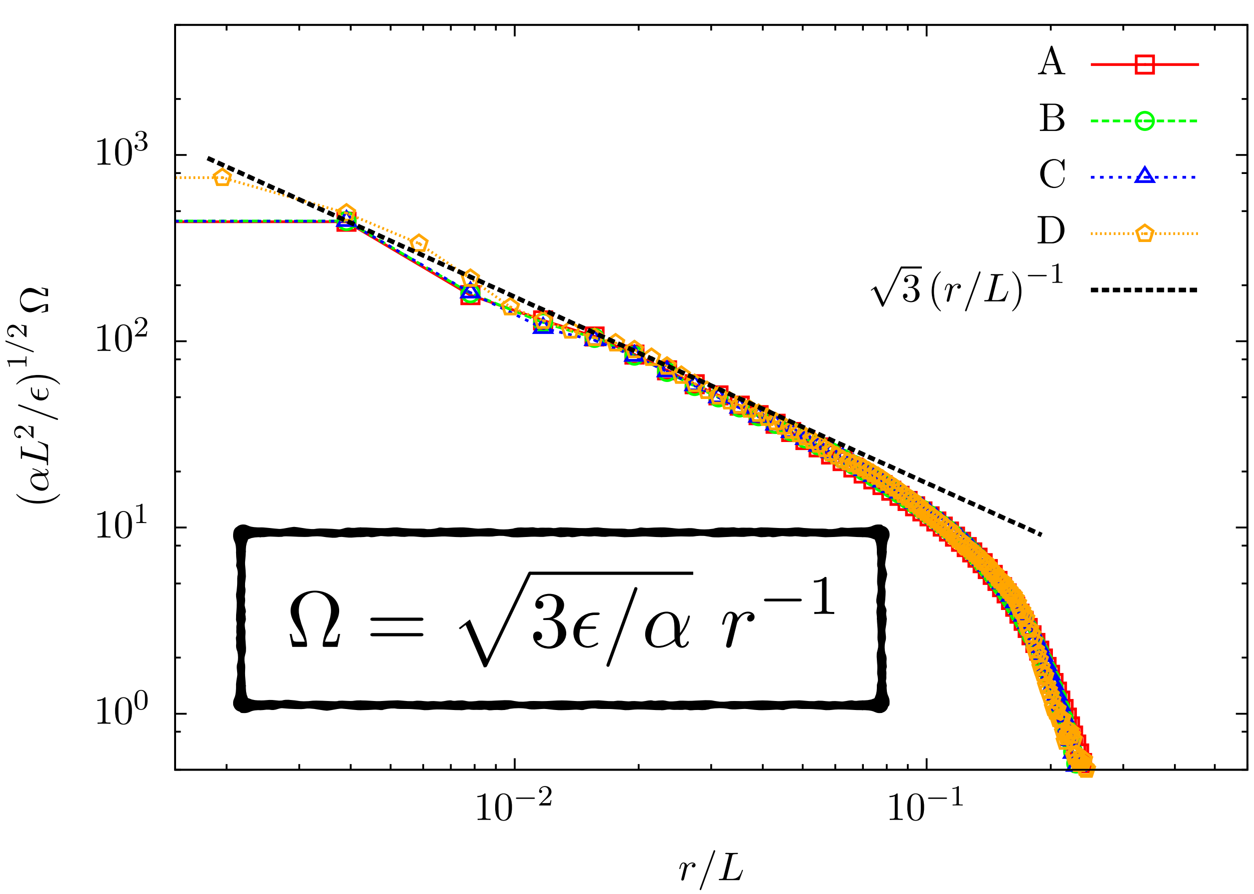
Numerical simulations

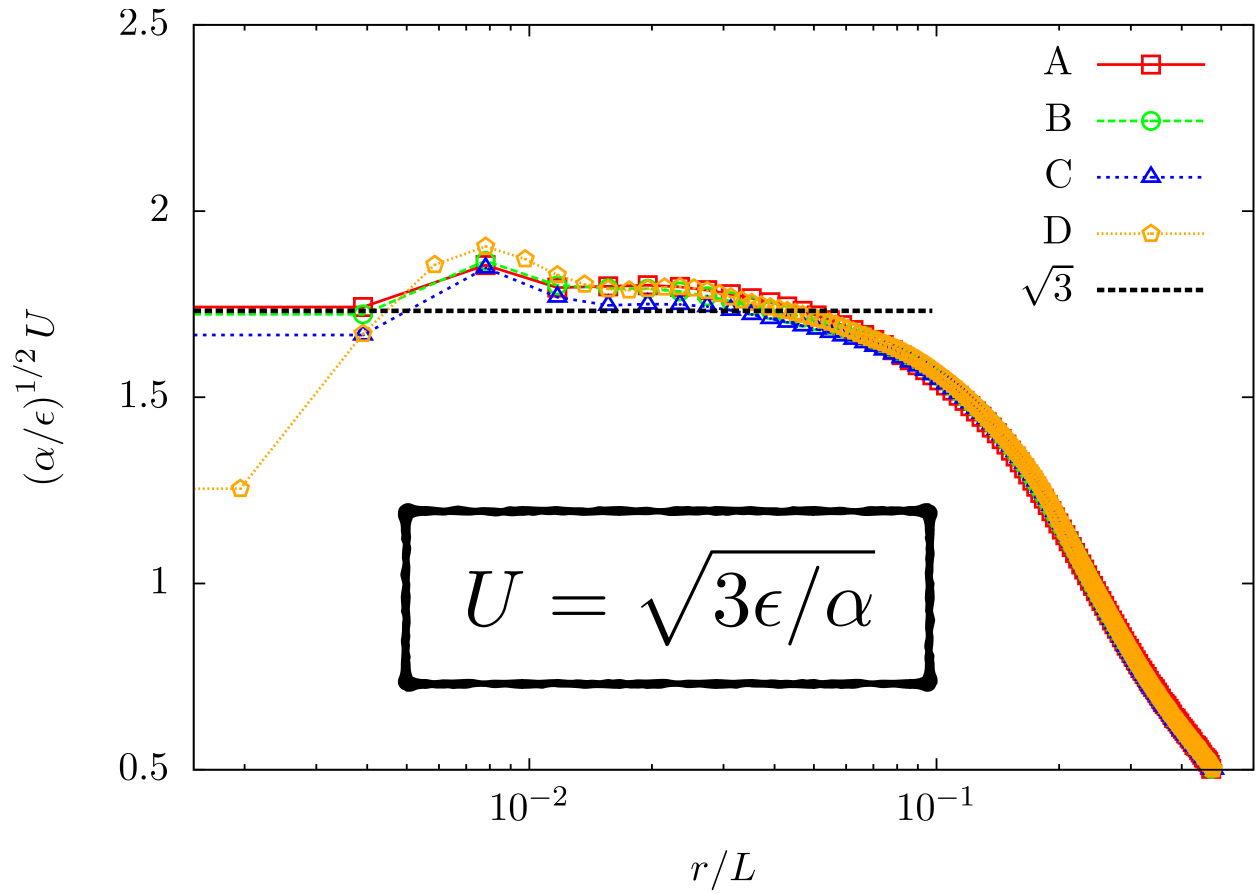
- Forced/dissipated pseudo-spectral simulations with small-scale forcing
- Simulations A-C have spatial resolution 512^2, while simulation D is 1024^2
- All simulations have different linear friction coefficient α



Mean vorticity profile



Mean azimuthal velocity profile



JL *et al.* Phys. Rev. Lett. **113**, 254503, (2014)

Condensates in rectangular domains

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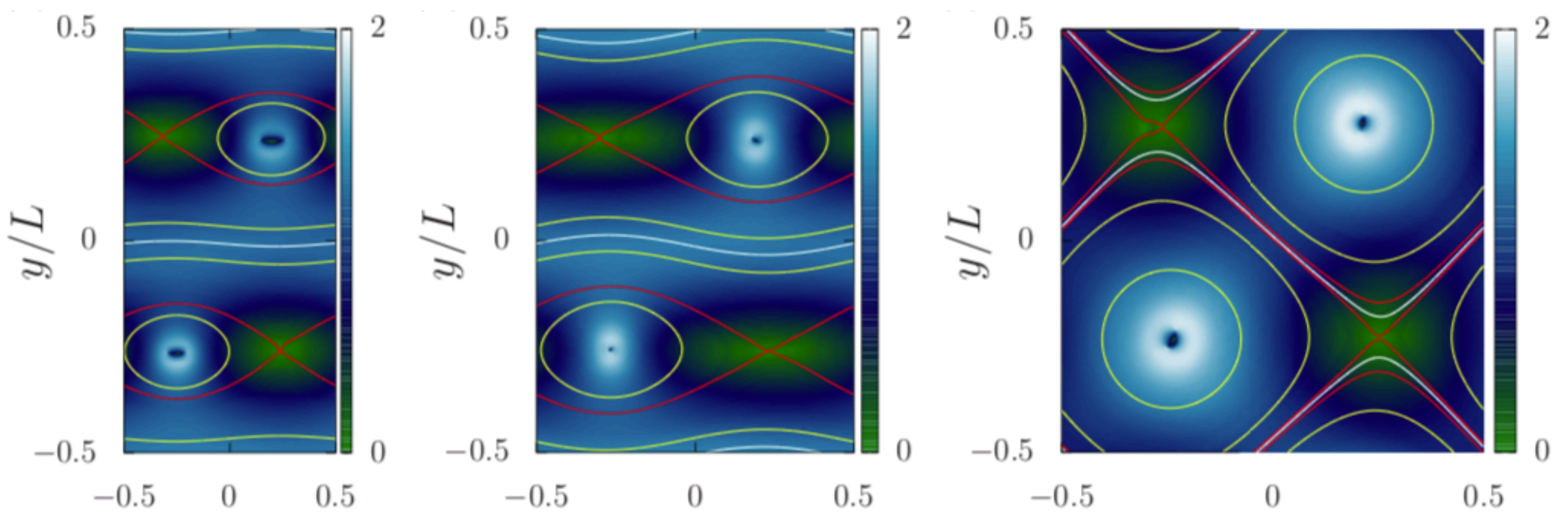
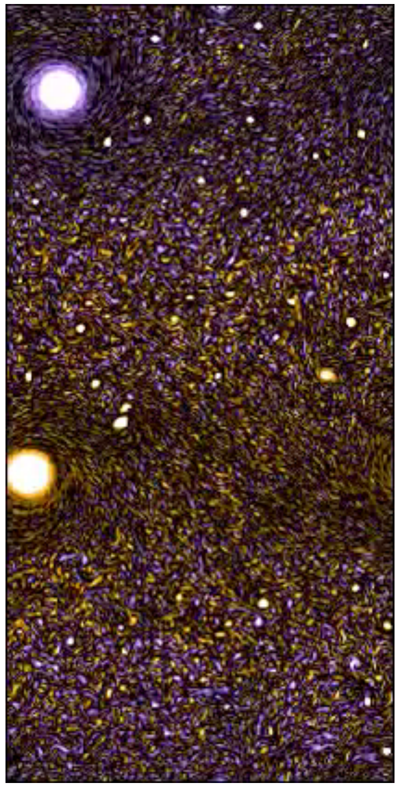
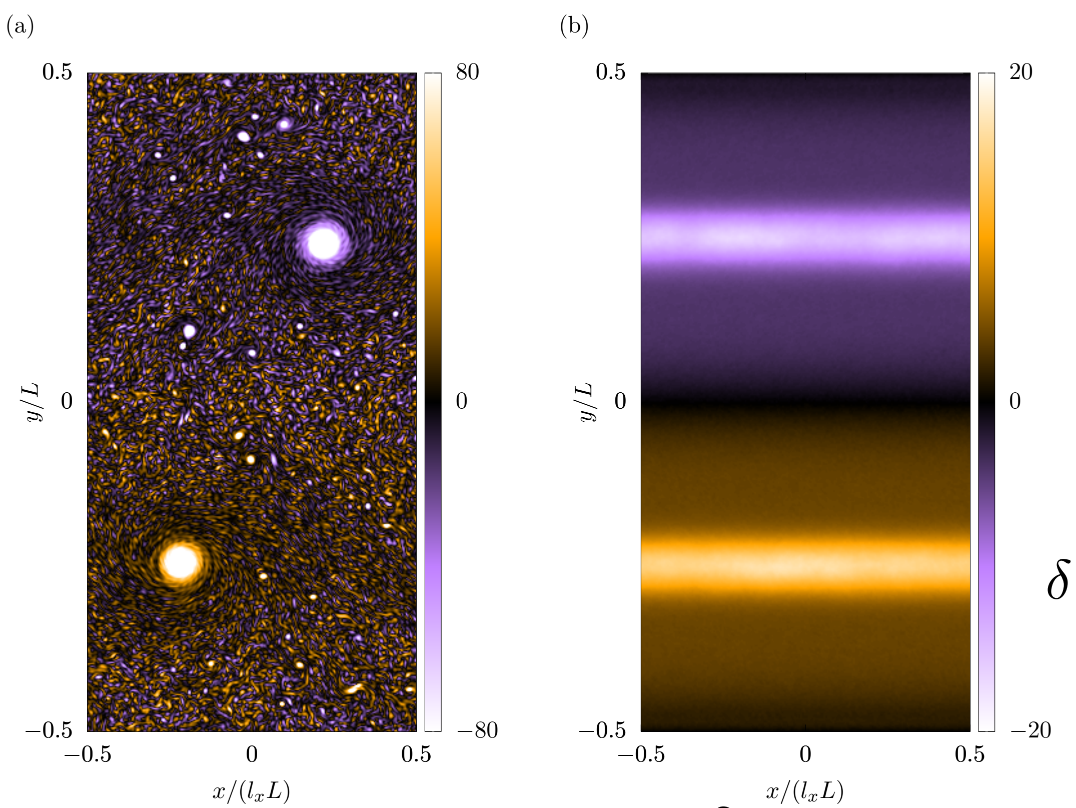
Energy condensates in rectangular domains

A. Frishman, JL, G. Falkovich, Phys. Rev. Fluids, 2, 032602, (2017)

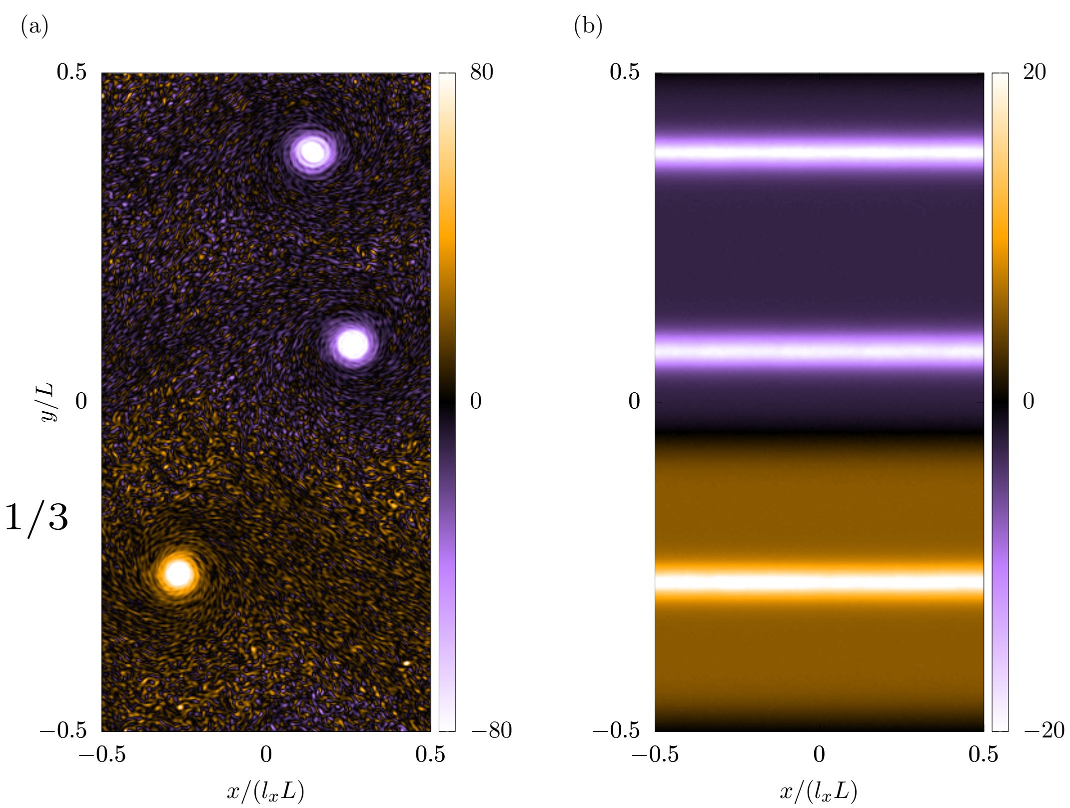




Temporal mean displays zonal symmetry



$$\delta = \left(\frac{\alpha^3}{\epsilon L^2}\right)^{1/3}$$



$\delta = 1.1 \times 10^{-2}$

$\delta = 2.8 \times 10^{-3}$

Zonal jet profile

Momentum and energy balance for zonal state

- Both balance equations imply any solution must satisfy

$$\boxed{\partial_y U \langle uv \rangle = \epsilon \qquad \partial_y \langle uv \rangle = -\alpha U}$$

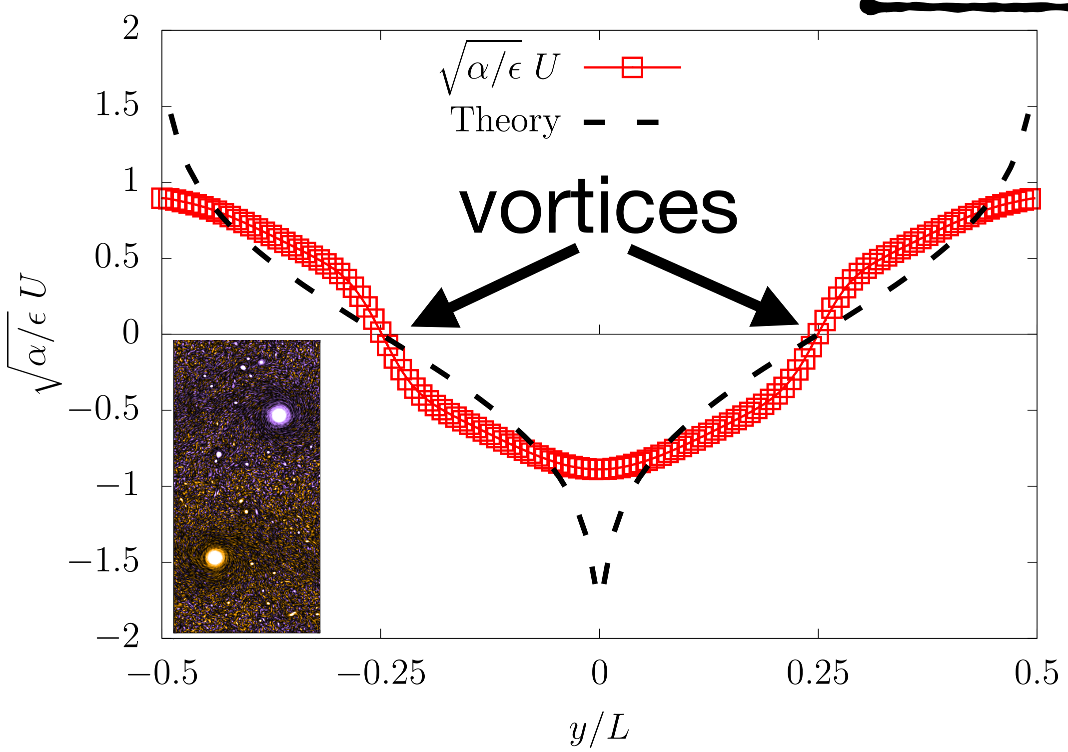
which cannot be satisfied when $\partial_y U \approx 0$ because $\langle uv \rangle$ must remain finite

A. Frishman, JL, G. Falkovich, Phys. Rev. Fluids, 2, 032602, (2017)

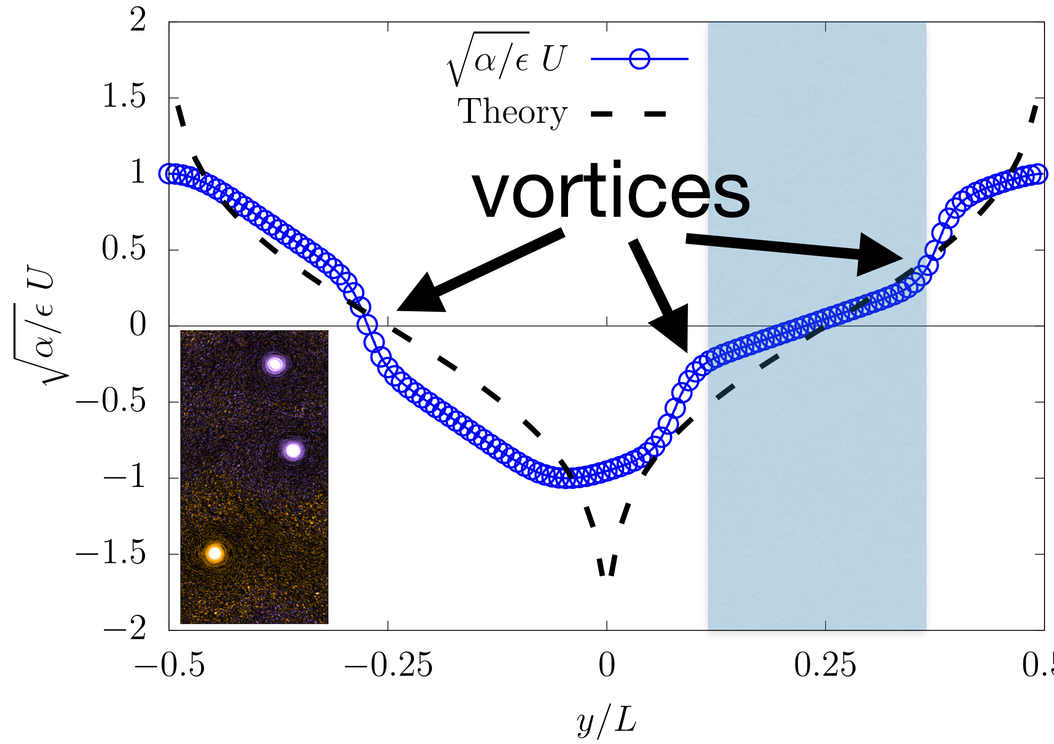
The closure cannot remain valid in the whole domain

Jet profile prediction

$$\boxed{U(y) = \sqrt{\frac{2\epsilon}{\alpha}} \mathrm{InvErf}\left(\frac{2y}{\pi}\right)}$$



Assumptions satisfied



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Thin Layer Turbulence:
2D to 3D Transition

Phenomenology of quasi-2D flows

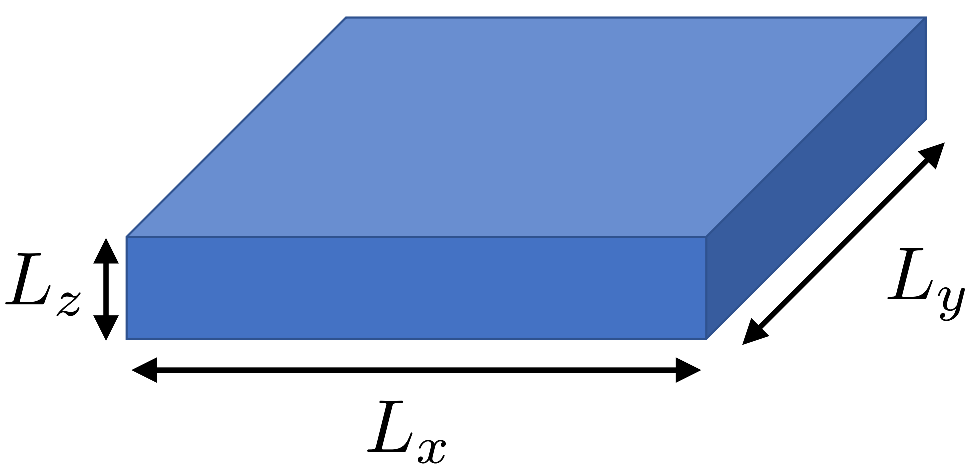
- Most real *2D flows* are **quasi-2D,** e.g. the height of Earth's atmosphere is ~100km, while the circumference is ~40,000km

3D Navier-Stokes equations

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\,\mathbf{v} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{v} + \mathbf{f}$$

$$\nabla\cdot\mathbf{v} = 0$$

Transition from 2D to 3D turbulence as thickness L_z increases



$0 < L_z < l_\nu < L_f < L_x, L_y$ 2D inverse cascade

$0 < l_\nu < L_z < L_f < L_x, L_y$?

$0 < l_\nu < L_f < L_z < L_x, L_y$ 3D direct cascade

Celani et al. Phys. Rev. Lett. **104**, 184506, (2010)
Musacchio and Boffetta, Phys. Fluids, **29**, 111106, (2017)
Musacchio and Boffetta, Phys. Rev. Fluids, **4**, 022602(R), (2019)

Energy growth as 2D becomes 3D

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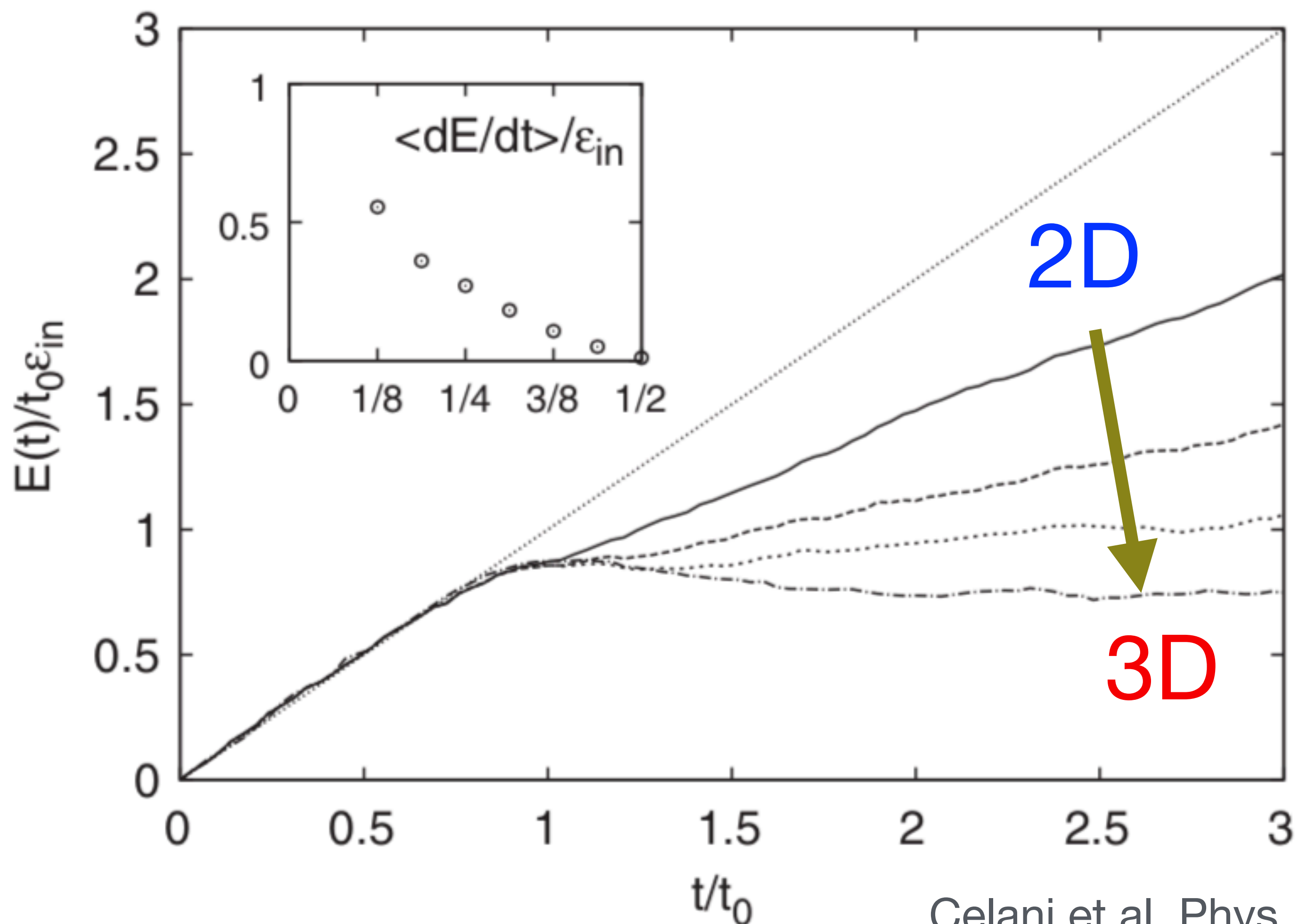
Numerical simulations of quasi-2D turbulence

- In pure 2D, energy grows **linear in time**
- The **energy growth rate decreases as the thickness increases**



Celani et al. Phys. Rev. Lett. **104**, 184506, (2010)

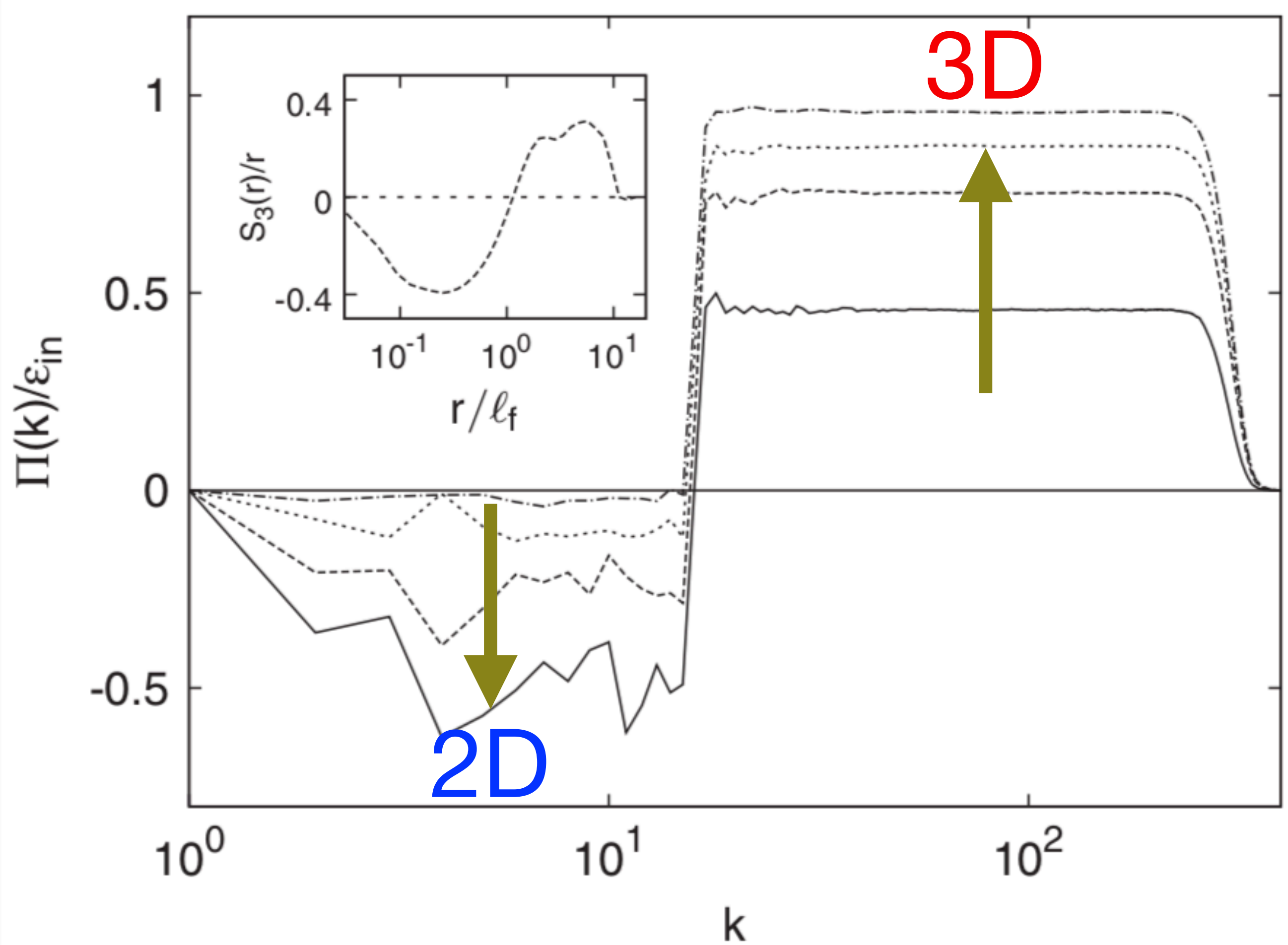
- For $L_z < l_\nu$ observed energy growth rate = rate of energy injection
- For $l_\nu < L_f/2 < L_z$ energy growth rate vanishes

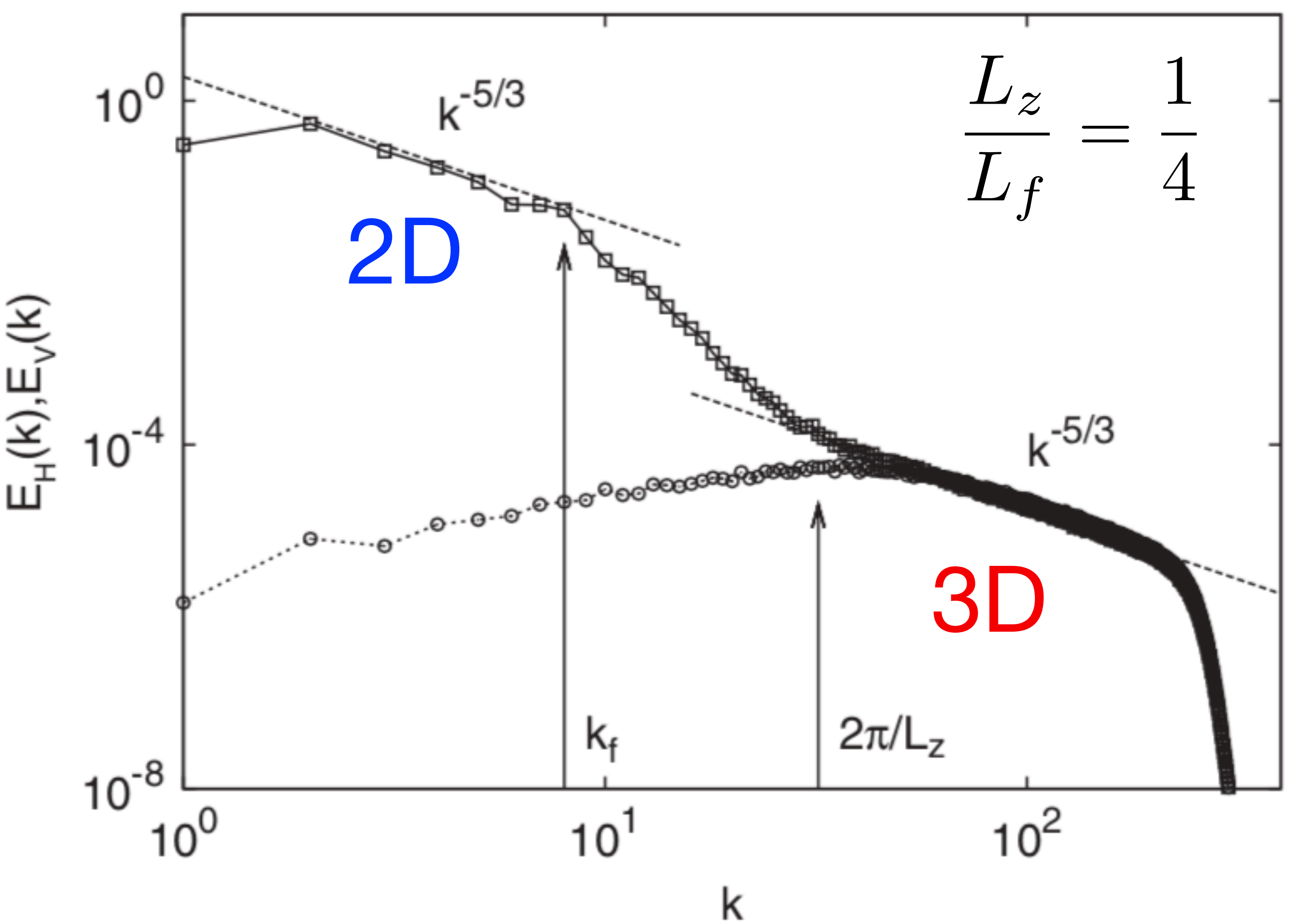
Split energy cascade

Coexisting inverse and direct energy cascades

- In the transition region, part of the **energy is transferred to large-scales via a 2D inverse cascade**, while the rest goes to **small-scales via a 3D direct cascade**

$$0 < l_\nu < L_z < L_f < L_x$$





Celani et al. Phys. Rev. Lett. **104**, 184506, (2010)

Summary

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- 2D Navier-Stokes is the **simplest model** for geophysical flows
- 2D turbulence is a **dual-cascade system**
- The inverse energy cascade leads to **energy condensation**
- **Mathematical Prediction** of large-scale mean flows
- **Quasi 2D turbulence**: 2D - 3D transition
- What I didn't mention: **conformal invariance**; **bistability; 2D geophysical turbulence**;…