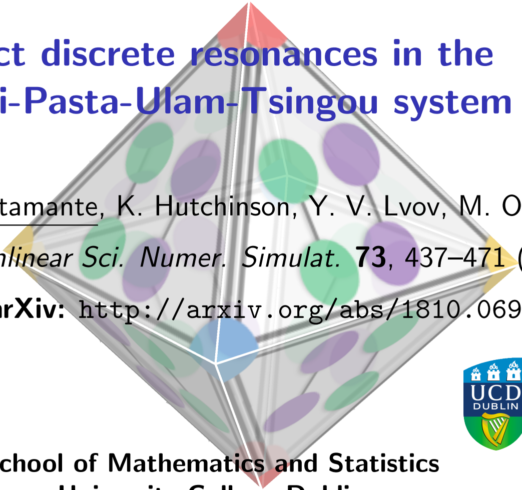
Exact discrete resonances in the Fermi-Pasta-Ulam-Tsingou system



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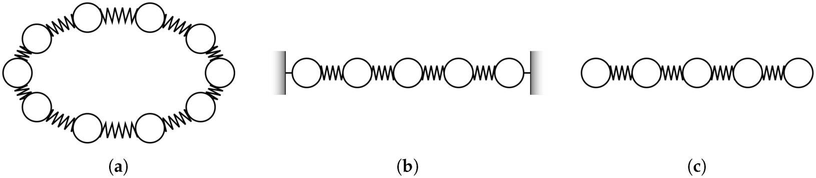
Fermi-Pasta-Ulam-Tsingou system

- N identical masses connected by anharmonic springs moving in one dimension. Studied in 1953, using numerical simulations (MANIAC).
- System did not relax to equilibrium; rather, a recurrence behaviour was observed.
- The result sparked the research field of nonlinear science: integrable systems such as Korteweg de Vries were related to this problem.



(Credits: A. L. Burin et al., Entropy **2019**, *21*(1), 51)

Hamiltonian for the $(\alpha + \beta)$ FPUT model

The Hamiltonian for a chain of N identical particles of mass m, connected by identical anharmonic springs, can be expressed as an unperturbed Hamiltonian, H_0, plus two perturbative terms, H_3, H_4:

$$H = H_0 + H_3 + H_4 \tag{1}$$

with

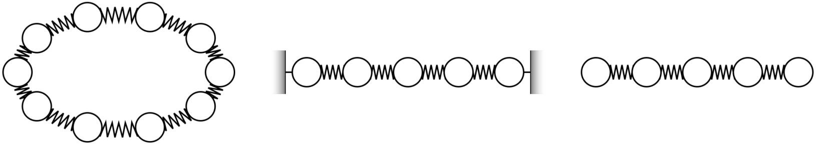
$$\begin{aligned} H_0 &= \sum_{j=1}^{N} \left(\frac{1}{2m} p_j^2 + \kappa \frac{1}{2} (q_j - q_{j+1})^2 \right), \\ H_3 &= \frac{\alpha}{3} \sum_{j=1}^{N} (q_j - q_{j+1})^3, \\ H_4 &= \frac{\beta}{4} \sum_{j=1}^{N} (q_j - q_{j+1})^4. \end{aligned} \tag{2}$$

$q_j(t)$ is the displacement of the particle j from its equilibrium position and $p_j(t)$ is the associated momentum.

Equations of motion for the original variables $q_j(t)$

$$
\begin{aligned}
m\ddot{q}_j &= \kappa(q_{j+1} + q_{j-1} - 2q_j) \\
&+ \alpha\big[(q_{j+1} - q_j)^2 - (q_j - q_{j-1})^2\big] \\
&+ \beta\big[(q_{j+1} - q_j)^3 - (q_j - q_{j-1})^3\big], \qquad j = 0, \dots, N-1
\end{aligned}
$$

This is known as the $\alpha + \beta$-FPUT model.



(a) (b) (c)

$q_0 = q_N$ $\quad$ $q_{-1} = q_N = 0$ $\quad$ $q_{-1} = q_0,\ q_{N-1} = q_N$

We will consider **periodic boundary conditions** from here on.

Equations in Fourier space: modular momentum condition

$$Q_k = \frac{1}{N}\sum_{j=0}^{N-1} q_j e^{-i2\pi kj/N}, \; P_k = \frac{1}{N}\sum_{j=0}^{N-1} p_j e^{-i2\pi kj/N},$$

$$\begin{aligned}
\frac{H}{N} &= \frac{P_0^2}{2m} + \frac{1}{2m}\sum_{k=1}^{N-1}\left(|P_k|^2 + m^2\,\omega_k^2\,|Q_k|^2\right) \\
&+ \frac{1}{3}\sum_{k_1,k_2,k_3=1}^{N-1} \tilde{V}_{1,2,3}\,Q_1Q_2Q_3\,\delta_{1+2+3} \\
&+ \frac{1}{4}\sum_{k_1,k_2,k_3,k_4=1}^{N-1} \tilde{T}_{1,2,3,4}\,Q_1Q_2Q_3Q_4\,\delta_{1+2+3+4},
\end{aligned}$$

Dispersion relation:

$$\omega_k = \omega(k) = 2\sqrt{\frac{\kappa}{m}}\sin(\pi k/N)\,, \quad 1 \leqslant k \leqslant N-1 \qquad (3)$$

$\delta_{1+2+3} = \delta(k_1 + k_2 + k_3 \mod N)$, (Kronecker δ), leading to the modular-arithmetic condition $k_1 + k_2 + k_3 = 0 \mod N$.

Equations of motion in Fourier space

The equations of motion take the following form:

$$\ddot{Q}_1 + \omega_1^2 Q_1 = \frac{1}{m} \sum_{k_2,k_3} \tilde{V}_{1,2,3} Q_2 Q_3 \delta_{1+2+3} + \frac{1}{m} \sum_{k_2,k_3,k_4} \tilde{T}_{1,2,3,4} Q_2 Q_3 Q_4 \delta_{1+2+3+4},$$

where all the sums on k_j go from 1 to $N-1$.

These are exact equations, representing **perturbed harmonic oscillators.**

Normal modes: introduced to diagonalise the Hamiltonian

$$Q_k = \frac{1}{\sqrt{2m\omega_k}} (a_k + a^*_{N-k}).$$

Equations of motion for the normal modes:

$$i\frac{\partial a_1}{\partial t} = \omega_{k_1} a_1 + \sum_{k_2,k_3} \left(V_{123} a_2 a_3 \delta_{1-2-3} + W_{123} a_2^* a_3 \delta_{1+2-3} + Z_{123} a_2^* a_3^* \delta_{1+2+3}\right)$$

$$+ \sum_{k_2,k_3,k_4} \Big(R_{1234}\, a_2 a_3 a_4 \delta_{1-2-3-4} + S_{1234}\, a_2^* a_3 a_4 \delta_{1+2-3-4}$$

$$+ T_{1234}\, a_2^* a_3^* a_4 \delta_{1+2+3-4} + U_{1234}\, a_2^* a_3^* a_4^* \delta_{1+2+3+4}\Big)\,.$$

Dynamical systems approach

$$i\frac{\partial a_1}{\partial t} = \omega_{k_1} a_1 + \sum_{k_2,k_3} \left(V_{123} a_2 a_3 \delta_{1-2-3} + W_{123} a_2^* a_3 \delta_{1+2-3} + Z_{123} a_2^* a_3^* \delta_{1+2+3}\right)$$

$$+ \sum_{k_2,k_3,k_4} \left(R_{1234}\, a_2 a_3 a_4 \delta_{1-2-3-4} + S_{1234}\, a_2^* a_3 a_4 \delta_{1+2-3-4} + T_{1234}\, a_2^* a_3^* a_4 \delta_{1+2+3-4} + U_{1234}\, a_2^* a_3^* a_4^* \delta_{1+2+3+4}\right).$$

There are three interesting regimes:

- Weakly nonlinear regime: amplitudes are small, so higher-order terms are small. Exact resonances dominates. **Normal form theory.** (Poincaré, Birkhoff, Arnold, etc.) [Bustamante et al., CNSNS **73**, 437 (2019)]
- Finite amplitudes: the terms of different orders are comparable. Bifurcations and chaos dominate. **Precession resonance.** [Bustamante et al., PRL **113**, 084502 (2014)] (cf. Critical Balance).
- Large amplitudes: the higher order term dominates. System recovers re-scaling symmetries. **Synchronisation of phases.** [Murray & Bustamante, JFM **850**, 624 (2018)]

Weakly nonlinear regime: Dominated by exact resonances

$$i\frac{\partial a_1}{\partial t} = \omega_{k_1} a_1 + \sum_{k_2,k_3} \left(V_{123} a_2 a_3 \delta_{1-2-3} + W_{123} a_2^* a_3 \delta_{1+2-3} + Z_{123} a_2^* a_3^* \delta_{1+2+3}\right)$$

$$+ \sum_{k_2,k_3,k_4} \left(R_{1234}\, a_2 a_3 a_4 \delta_{1-2-3-4} + S_{1234}\, a_2^* a_3 a_4 \delta_{1+2-3-4}\right.$$

$$\left. + T_{1234}\, a_2^* a_3^* a_4 \delta_{1+2+3-4} + U_{1234}\, a_2^* a_3^* a_4^* \delta_{1+2+3+4}\right).$$

In the limit of small amplitudes, the only relevant interactions are those interactions between wavenumbers that satisfy the momentum equation

$$k_1 \pm k_2 \pm \ldots \pm k_M = 0 \pmod N$$

and the frequency resonance equation

$$\sin(\pi k_1/N) \pm \ldots \pm \sin(\pi k_M/N) = 0.$$

These are called M-wave resonances, where $M \geqslant 3$ is an integer.

- The unknowns are the integers $1 \leqslant k_1, \ldots, k_M \leqslant N - 1$.
- When $M = 3$ there are no solutions to these equations.
- Therefore one can eliminate those interactions via a near-identity transformation.

Normal form variables: Near-identity transformation

$$a_1 = b_1 + \sum_{k_2,k_3} \left(A^{(1)}_{1,2,3} b_2 b_3 \delta_{1-2-3} + A^{(2)}_{1,2,3} b_2^* b_3 \delta_{1+2-3} + A^{(3)}_{1,2,3} b_2^* b_3^* \delta_{1+2+3} \right) +$$

$$+ \sum_{k_2,k_3,k_4} (B^{(1)}_{1,2,3,4} b_2 b_3 b_4 \delta_{1-2-3-4} + B^{(2)}_{1,2,3,4} b_2^* b_3 b_4 \delta_{1+2-3-4} +$$

$$B^{(3)}_{1,2,3,4} b_2^* b_3^* b_4 \delta_{1+2+3-4} + B^{(4)}_{1,2,3,4} b_2^* b_3^* b_4^* \delta_{1+2+3+4}) + \ldots$$

and select the matrices $A^{(i)}_{1,2,3}$, $B^{(i)}_{1,2,3,4}$ in order to remove non-resonant interactions.
For example, the choice

$$A^{(1)}_{1,2,3} = \frac{V_{1,2,3}}{\omega_3 + \omega_2 - \omega_1},\ A^{(2)}_{1,2,3} = \frac{2V_{1,2,3}}{\omega_3 - \omega_2 - \omega_1},\ A^{(3)}_{1,2,3} = \frac{V_{1,2,3}}{-\omega_3 - \omega_2 - \omega_1}.$$

eliminates the 3-wave interactions, leading to a system of equations for the normal form variables $b_1, \ldots, b_{N-1}$ (next slide):

Normal form equations of motion

$$i\frac{\partial b_1}{\partial t} = \omega_{k_1} b_1 + \sum_{k_2,k_3,k_4} \big(R_{1234}\, b_2 b_3 b_4 \delta_{1-2-3-4} + S_{1234}\, b_2^* b_3 b_4 \delta_{1+2-3-4}$$
$$+ T_{1234}\, b_2^* b_3^* b_4 \delta_{1+2+3-4} + U_{1234}\, b_2^* b_3^* b_4^* \delta_{1+2+3+4}\big) \quad + \mathcal{O}(|b|^5)$$

- **Does the transformation converge?** Open question in general. See "On the convergence of the normal form transformation in discrete Rossby and drift wave turbulence" by Walsh & Bustamante, arXiv:1904.13272
- **Can we eliminate some of the 4-wave interactions?** Yes, provided they are not resonant.
- **The transformation created extra interactions:** 5-wave, 6-wave, etc. Therefore the question about resonances is relevant for all possible M waves.

FPUT exact resonances: Diophantine equations

Definition (M-wave resonance)

Let N be the number of particles of the FPUT system. An M-wave resonance is a list (i.e., a multi-set) $\{k_1, \ldots, k_S; k_{S+1}, \ldots, k_{S+T}\}$ with $S, T > 0$, $S + T = M$ and $1 \leqslant k_j \leqslant N - 1$ for all $j = 1, \ldots, M$, that is a solution of the momentum conservation and frequency resonance conditions

$$k_1 + \ldots + k_S = k_{S+1} + \ldots + k_{S+T} \pmod N,$$
$$\omega(k_1) + \ldots + \omega(k_S) = \omega(k_{S+1}) + \ldots + \omega(k_{S+T}),$$

where $\omega(k) = 2\sin(\pi k/N)$.

Physically, this corresponds to the conversion process of S waves into T waves. The Hamiltonian term is proportional to $b_{k_1} \cdots b_{k_S} b^*_{k_{S+1}} \cdots b^*_{k_{S+T}}$.

Preliminary: Forbidden M-wave resonances

Theorem (Forbidden Processes)

Resonant processes converting 1 *wave to* $M-1$ *waves or* $M-1$ *waves to* 1 *wave do not exist, for any* $M \neq 2$. *Also, resonant processes converting* 0 *wave to* M *waves or* M *waves to* 0 *wave do not exist, for any* $M > 0$.

Proof. The function $\omega(k) = 2|\sin(\pi k/N)|$ is strictly subadditive for $k \in \mathbb{R}$, $k \notin N\mathbb{Z}$:

$$\omega(k_1 + k_2) < \omega(k_1) + \omega(k_2), \quad k_1, k_2 \in \mathbb{R}\backslash N\mathbb{Z}\,.$$

Therefore, for example, resonant processes converting 2 waves into 1 wave (or vice versa) are not allowed because this would require $\omega(k_1 + k_2) = \omega(k_1) + \omega(k_2)$, which is not possible. Similarly, resonant processes converting $M-1$ waves into 1 wave (or vice versa) are not allowed because subadditivity implies

$$\omega(k_1 + \ldots + k_p) < \omega(k_1) + \ldots + \omega(k_p), \quad k_1, \ldots, k_p \in \mathbb{R}\backslash N\mathbb{Z},$$

for any $p \geqslant 2$. □

What is new about 4-wave, 5-wave and 6-wave resonances?

- New methods to construct M-wave resonances: **Pairing-off method** & **Cyclotomic method.**
- The appropriate method to be used depends on properties of the number of particles N and the number of waves M.
- The case of 4-**wave resonances** has been studied extensively and all solutions are known.
- 4-**wave resonances are integrable** and thus they do not produce energy mixing across the Fourier spectrum: one needs to go to higher orders.
- The case of 5-**wave resonances** is completely new and relies on the existence of cyclotomic polynomials (to be defined below).
- The case of 6-**wave resonances** is also new and both methods (pairing-off and cyclotomic) are used to construct them.
- We do not need to search for M-wave resonances with $M > 6$ because these will provide less relevant corrections to the system's behaviour.

Pairing-off method to obtain $2S$-wave resonances converting S waves to S waves (for any N)

$$k_1 + \ldots + k_S = k_{S+1} + \ldots + k_{S+S} \pmod{N},$$
$$\omega(k_1) + \ldots + \omega(k_S) = \omega(k_{S+1}) + \ldots + \omega(k_{S+S}),$$

- Due to the identities $\omega(k) = \omega(N-k), \quad k = 1, \ldots, N-1$, one can "pair-off" incoming and outgoing waves, as follows:

  $$k_{S+j} = N - k_j\,, \qquad j = 1, \ldots, S.$$

  In this way, the frequency resonance condition is automatically solved "by pairs" since $\omega(k_{S+j}) = \omega(k_j)$.
- The momentum conservation condition leads to a single equation:

  $$k_1 + \ldots + k_S = \frac{N\nu}{2}\,,$$

  where the integer variable ν satisfies $2S/N \leqslant \nu < 3S/2$ and is introduced as a parameterisation of the momentum conservation condition.

The need for a more general method illustrated with the case $N = 6$

- When the number of particles N is an **odd prime or a power of** 2, any M-wave resonance must be of pairing-off form, and in particular M must be even.
- In the case when N (number of particles) is arbitrary, the pairing-off solutions to the $2L$-wave resonant conditions (with $2L \geqslant 6$) do not exhaust all possible solutions.
- For example, in the case $N = 6$ (six particles) there is a 6-wave resonance that is not of pairing-off form:

$$1{+}1{+}5{+}5 = 3{+}3 \pmod 6\,, \quad \omega(1){+}\omega(1){+}\omega(5){+}\omega(5) = \omega(3){+}\omega(3)\,.$$

The frequency resonance condition is satisfied because of the identity

$$\omega(1) + \omega(5) = \omega(3), \quad \text{or} \quad \sin \pi/6 + \sin 5\pi/6 = \sin 3\pi/6\,,$$

which is reminiscent of a triad resonance. **What is the origin of this identity?** The answer is given in terms of the $2N$-th root of unity and the so-called cyclotomic polynomials.

Writing the resonance conditions in terms of real polynomials on the $(2N)^{\mathrm{th}}$ root of unity

We write the dispersion relation as a complex exponential:

$$\omega(k) \equiv 2\sin(\pi k/N) = -i\left(\zeta^{k} - \zeta^{-k}\right),$$

where

$$\zeta = \exp\left(\frac{i\,\pi}{N}\right)$$

is a primitive $2N$-th root of unity: $\zeta^{2N} = 1$.
In terms of ζ, the frequency resonance condition is

$$(\zeta^{k_1} - \zeta^{-k_1}) + \ldots + (\zeta^{k_S} - \zeta^{-k_S}) = (\zeta^{k_{S+1}} - \zeta^{-k_{S+1}}) + \ldots + (\zeta^{k_{S+T}} - \zeta^{-k_{S+T}}).$$

Recalling that ζ is a unit complex number, it follows that ζ^{-k_j} is the complex conjugate of ζ^{k_j}, so the above is equivalent to the statement that a polynomial is real:

$$\rho(\zeta) \equiv \zeta^{k_1} + \ldots + \zeta^{k_S} - (\zeta^{k_{S+1}} + \ldots + \zeta^{k_{S+T}}) \in \mathbb{R}.$$

In other words, **solving the frequency resonance conditions is an easy task:** it amounts to finding all real polynomials on the variable ζ.

$$\rho(\zeta) \equiv \zeta^{k_1} + \ldots + \zeta^{k_S} - (\zeta^{k_{S+1}} + \ldots + \zeta^{k_{S+T}}) \in \mathbb{R}\,, \qquad \zeta = \exp\left(\frac{i\,\pi}{N}\right)\,.$$

Example (Pairing-off resonances)

The pairing-off resonances correspond to a "pairing-off" real polynomial, made out of real binomials: by setting $S = T$ *and* $k_{S+j} = N - k_j$ *we obtain*

$$\begin{aligned}\rho(\zeta) &= \zeta^{k_1} + \ldots + \zeta^{k_S} - \left(\zeta^{N-k_1} + \ldots + \zeta^{N-k_S}\right) \\ &= \left(\zeta^{k_1} + \zeta^{-k_1}\right) + \ldots + \left(\zeta^{k_S} + \zeta^{-k_S}\right)\,,\end{aligned}$$

which is real, pair by pair.

$$\rho(\zeta) \equiv \zeta^{k_1} + \ldots + \zeta^{k_S} - (\zeta^{k_{S+1}} + \ldots + \zeta^{k_{S+T}}) \in \mathbb{R}\,, \qquad \zeta = \exp\left(\frac{i\,\pi}{N}\right)\,.$$

Example (Cyclotomic resonance for $N = 6$)

This resonance corresponds to an element of the kernel of the above map, since, for $N = 6$ we have

$$\zeta = \exp\left(\frac{i\,\pi}{6}\right) \Longrightarrow \zeta + \zeta^5 - \zeta^3 = 0\,.$$

Thus, the resonance corresponds to $S = 4$, $T = 2$ with wavenumbers $k_1 = k_2 = 1, \quad k_3 = k_4 = 5$, and $k_5 = k_6 = 3$ so we obtain

$$\rho(\zeta) = \zeta^{k_1} + \zeta^{k_2} + \zeta^{k_3} + \zeta^{k_4} - \zeta^{k_5} - \zeta^{k_6} = 2\left(\zeta^{k_1} + \zeta^{k_3} - \zeta^{k_5}\right) = 0,$$

again real.

Momentum condition: Resonant FPUT polynomial

$$\rho(\zeta) \equiv \zeta^{k_1} + \ldots + \zeta^{k_S} - (\zeta^{k_{S+1}} + \ldots + \zeta^{k_{S+T}}) \in \mathbb{R}\,, \qquad \zeta = \exp\left(\frac{i\,\pi}{N}\right)\,.$$

The momentum condition is easily seen to be

$$\rho'(1) = 0 \pmod N.$$

- **Definition:** a resonant FPUT polynomial as a polynomial $\rho(x)$ of the above form, such that $\rho(\zeta)$ is real and such that $\rho'(1) = 0 \pmod N$.
- Knowing a resonant FPUT polynomial is equivalent to finding an M-wave resonance.
- The defining equations are linear so we want to find a “basis” for the resonant FPUT polynomials.

The cyclotomic method: Constructing resonant FPUT polynomials of short length (1/2)

$$\rho(\zeta) \equiv \zeta^{k_1} + \ldots + \zeta^{k_S} - (\zeta^{k_{S+1}} + \ldots + \zeta^{k_{S+T}}) \in \mathbb{R}\,, \qquad \zeta = \exp\left(\frac{i\,\pi}{N}\right)\,.$$

Theorem

Suppose $3|N$ (i.e., N is divisible by 3). Then the polynomials

$$f_n(x) = x^n - x^{n+N/3} + x^{n+2N/3}, \quad n = 1, \ldots, N/3 - 1$$

are real FPUT polynomials in the sense that $f_n(\zeta) = 0$.

- Notice that $\omega(n) - \omega(n + N/3) + \omega(n + 2N/3) = 0$, reminiscent of a resonant triad.
- We can add to this polynomial any pair-off polynomial. This produces 8 possible combinations. For example, adding $x^{N-n} - x^n$ gives

$$g_n(x) = x^{N-n} - x^{n+N/3} + x^{n+2N/3}, \quad n = 1, \ldots, N/3 - 1.$$

The cyclotomic method: Constructing resonant FPUT polynomials of short length (2/2)

$$\rho(\zeta) \equiv \zeta^{k_1} + \ldots + \zeta^{k_S} - (\zeta^{k_{S+1}} + \ldots + \zeta^{k_{S+T}}) \in \mathbb{R}\,, \qquad \zeta = \exp\left(\frac{i\,\pi}{N}\right)\,.$$

The momentum condition, $\rho'(1) = 0 \pmod N$, is not satisfied by polynomials with three terms. We need to add extra terms, again of pairing-off form, but which do not cancel:

Theorem

Suppose $3|N$ *(i.e.,* N *is divisible by* 3*). Then the polynomials*

$$f_{n,q}(x) = x^n - x^{n+N/3} + x^{n+2N/3} + x^q - x^{N-q}$$

are resonant FPUT polynomials if and only if

$$n + N/3 + 2q = 0 \pmod N.$$

Notice that there will be 8 versions of this theorem.

Octahedra ($3 \mid N$ and N odd): $n = 2, 4, \ldots, \frac{N}{3} - 1.$

$$\left\{n, \frac{2N}{3} + n, \frac{N}{3} - \frac{n}{2}; \frac{N}{3} + n, \frac{2N}{3} + \frac{n}{2}\right\}$$

$$\left\{n, \frac{N}{3} - n, \frac{n}{2}; \frac{N}{3} + n, N - \frac{n}{2}\right\}$$

$$\left\{n, \frac{2N}{3} + n, N - \frac{3n}{2}; \frac{2N}{3} - n, \frac{3n}{2}\right\}$$

$$\left\{n, \frac{N}{3} - n, \frac{2N}{3} - \frac{n}{2}; \frac{2N}{3} - n, \frac{N}{3} + \frac{n}{2}\right\}$$

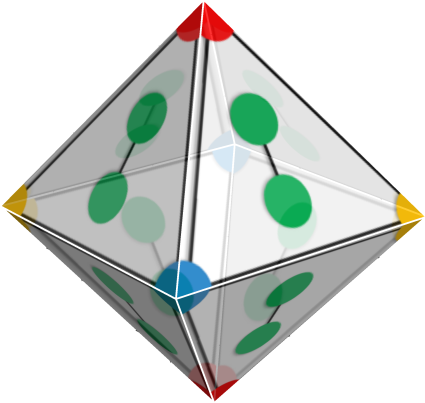
$$\left\{N - n, \frac{2N}{3} + n, \frac{N}{3} + \frac{n}{2}; \frac{N}{3} + n, \frac{2N}{3} - \frac{n}{2}\right\}$$

$$\left\{N - n, \frac{N}{3} - n, N - \frac{3n}{2}; \frac{N}{3} + n, \frac{3n}{2}\right\}$$

$$\left\{N - n, \frac{2N}{3} + n, N - \frac{n}{2}; \frac{2N}{3} - n, \frac{n}{2}\right\}$$

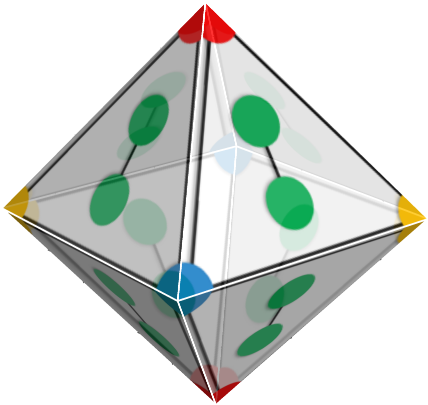
$$\left\{N - n, \frac{N}{3} - n, \frac{2N}{3} + \frac{n}{2}; \frac{2N}{3} - n, \frac{N}{3} - \frac{n}{2}\right\}$$

Octahedra ($3 \mid N$ and N odd): $n = 2, 4, \ldots, \frac{N}{3} - 1.$



- Each octahedron is a "cluster" of 14 nonlinearly interacting Fourier modes: $n/2, n, 3n/2, n/2 + N/3, n + N/3, n/2 + 2N/3, n + 2N/3$ and their "pair-off conjugates".

Octahedra ($3 \mid N$ and N odd): $n = 2, 4, \ldots, \frac{N}{3} - 1$.



- Divisibility: $n/2, n, 3n/2, n/2 + N/3, n + N/3, n/2 + 2N/3, n + 2N/3$ share the same divisors (apart from powers of 2 and 3).

Summary of clusters

$\mathbf{N > 6}$	$5 \nmid N$	$5 \mid N$
$3 \nmid N$	No Quintets.	No Quintets.
$3 \mid N \wedge 6 \nmid N$	$\lfloor \frac{N}{6} \rfloor$ Clusters: 8 Quintets Each; Total: $8\lfloor \frac{N}{6} \rfloor$ Quintets.	1 Extra Cluster: 2 Quintets; Total: $8\lfloor \frac{N}{6} \rfloor + 2$ Quintets.
$6 \mid N \wedge 12 \nmid N$	$\frac{1}{2}\left(\frac{N}{6} - 1\right)$ Clusters: 16 Quintets Each; Total: $8(\frac{N}{6} - 1)$ Quintets.	1 Extra Cluster: 2 Quintets; Total: $8(\frac{N}{6} - 1) + 2$ Quintets
$12 \mid N$	$\frac{N}{12} - 1$ Clusters: 16 Quintets Each; 1 Cluster: 6 Quintets; Total: $16(\frac{N}{12} - 1) + 6$ Quintets.	1 Extra Cluster: 2 Quintets; Total: $16(\frac{N}{12} - 1) + 6 + 2$ Qu

Table: *Summary of cases of 5-wave resonances for $N > 6$, regarding the counting of octahedron clusters and total number of quintets.*

Connectivity across Clusters: Superclusters. Example: $N = 75$

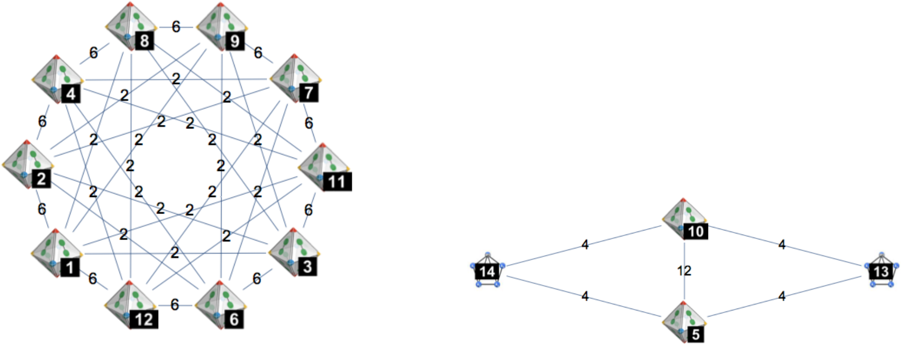


Figure: *Super-cluster $\mathcal{S}_{75}$ for $N = 3 \cdot 5^2 = 75$. All 10 vertices in the component $\mathcal{S}_{75}^{(1)}$ (left) have 14 wavenumbers each.* **The greatest common divisor amongst the wavenumbers in this component is 1.** *In the component $\mathcal{S}_{75}^{*}$ (right) the vertices numbered 5 and 10 have 12 wavenumbers each, which are strongly connected since their connecting edge has label 12. The vertices numbered 13 and 14 have 5 wavenumbers each.* **The greatest common divisor amongst the wavenumbers in this component is 5.**

The number of disjoint components depends on the number of divisors of N. Example: $N = 420$

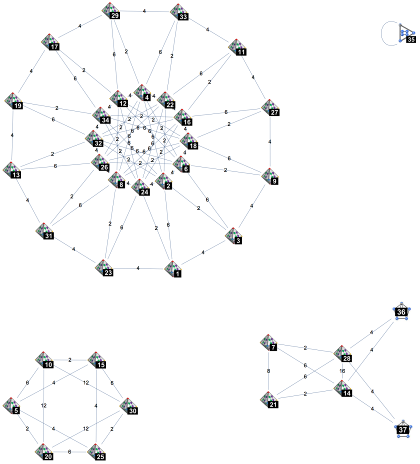


Figure: *Colour online. Super-cluster $\mathcal{S}_{420}$ for $N = 2^2 \cdot 3 \cdot 5 \cdot 7 = 420$. All 24 vertices in the component $\mathcal{S}_{420}^{(1)}$ (top left) and all 6 vertices in the component $\mathcal{S}_{420}^{(5)}$ (bottom left) have 22 wavenumbers each.* **The greatest common divisor amongst the wavenumbers in each component is: 1 (top left), 5 (bottom left), 7 (bottom right) and 35 (top right).**

Types of M-wave resonances as a function on N

$N \leqslant 6$	Lowest-order Resonances	Type of Resonance
3	6-wave	Pairing-off
4	4-wave & 6-wave	Pairing-off
5	6-wave	Pairing-off
6	4-wave & 6-wave	Pairing-off & Cycloton

$N > 6$	Lowest-order Resonances	Type of Resonance
$1 \pmod 6 \quad (7, 13, 19, \ldots)$	6-wave	Pairing-off
$2 \pmod 6 \quad (8, 14, 20, \ldots)$	4-wave & 6-wave	Pairing-off
$\mathbf{3} \pmod{\mathbf{6}} \quad (\mathbf{9}, \mathbf{15}, \mathbf{21}, \ldots)$	**$\mathbf{5}$-wave**	**Cyclotomic**
$4 \pmod 6 \quad (10, 16, 22, \ldots)$	4-wave & 6-wave	Pairing-off
$5 \pmod 6 \quad (11, 17, 23, \ldots)$	6-wave	Pairing-off
$\mathbf{0} \pmod{\mathbf{6}} \quad (\mathbf{12}, \mathbf{18}, \mathbf{24}, \ldots)$	4-wave & **$\mathbf{5}$-wave**	Pairing-off & **Cycloton**

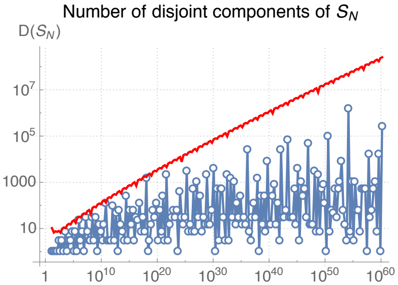
Table: *Study of lowest order of FPUT irreducible resonances that are not of Birkhoff normal form (in other words, resonances that effectively exchange energy amongst modes), as a function of N. The cases admitting 5-wave resonances are highlighted in* **boldface***. 4-wave resonances are always in so-called resonant Birkhoff normal form, which do not produce effective energy transfers throughout the whole spectrum of modes. In contrast, 5- and 6-wave irreducible resonances cannot be simplified in terms of resonant Birkhoff normal forms, because they mix energies over a wide range of modes.*

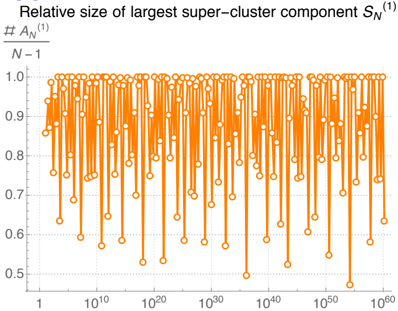
Summary of exact results

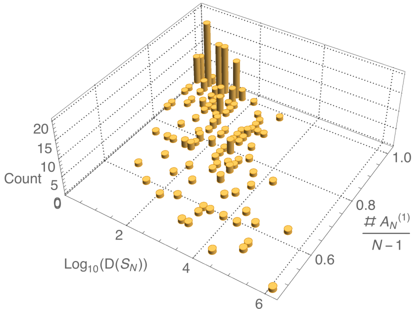
Considering $N \geqslant 6$,

- When N is prime or a power of 2, only pairing-off resonances exist, and they transform S waves into S waves.
- The cyclotomic method allows for the explicit construction of 5-wave resonances when N is divisible by 3.
- 4-wave resonances lead to disjoint clusters, for any N.
- **5-wave resonances are inter-connected in octahedra (connection is via common modes).**
- **These octahedra are further connected into superclusters (connection is via common modes).**
- **The number of disjoint superclusters is roughly equal to the number of divisors of N which are not divisible by 3 or 2.**
- 6-wave resonances exist for any N. They lead to one big interconnected cluster.

Sensitivity of 5-wave superclusters with respect to N: Does the $N \to \infty$ limit make sense?

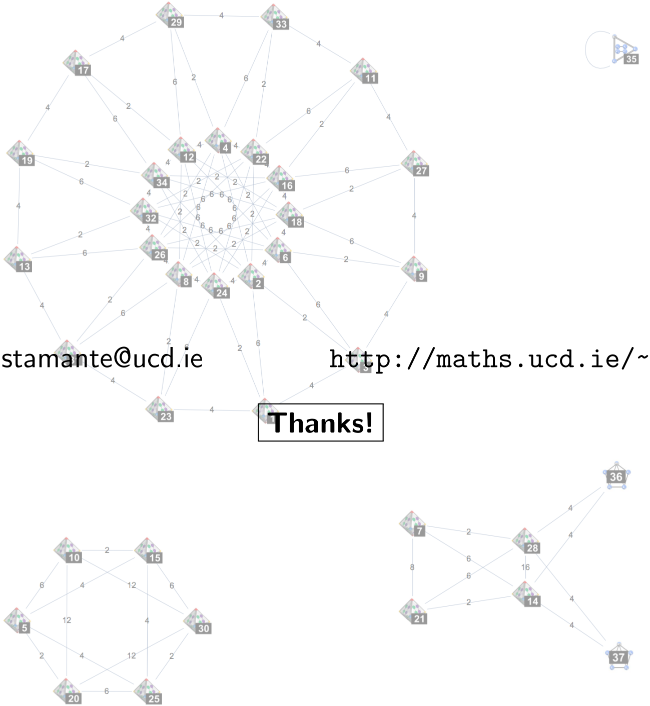






Looking forward: Thermalisation

- 4-wave resonances alone do not produce thermalisation.
- The divisibility of N could play an important role: if 5-waves dominated, we would obtain a different scaling for thermalisation time with respect to 6-wave dominated thermalisation.
- Are superclusters well connected enough to allow thermalisation via 5-wave resonances?
- Boundary conditions are important: for fixed (or free) boundary conditions, there are simply no resonances! (For any N).
- Convergence of the normal form transformation should be investigated.



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Thanks!