Wave Turbulence in Nonlinear Optics, BECs, and Related Areas
Aston University, Birmingham
12 Dec, 2017

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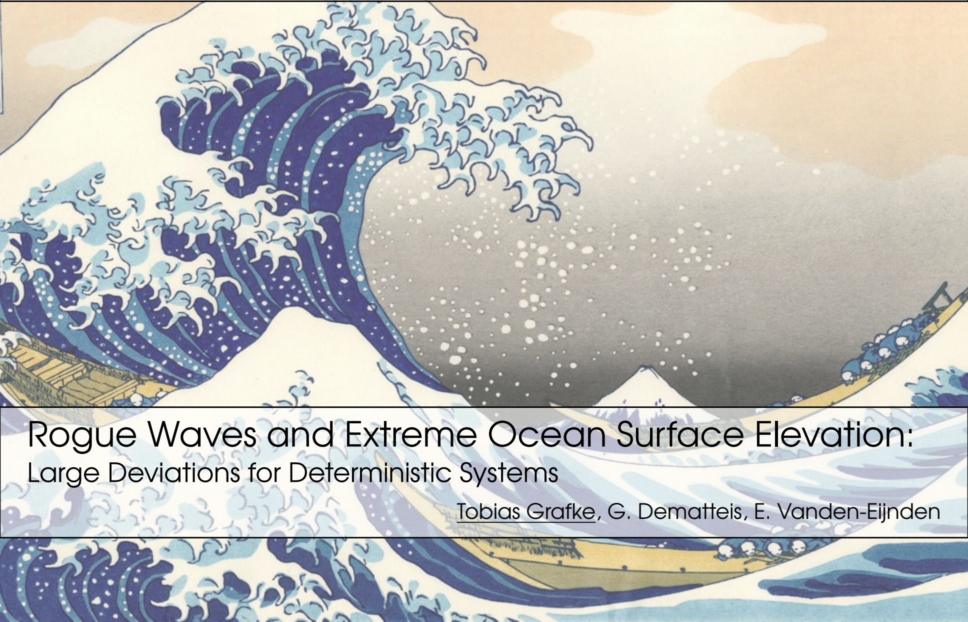


Rogue Waves and Extreme Ocean Surface Elevation:

Large Deviations for Deterministic Systems

Tobias Grafke, G. Dematteis, E. Vanden-Eijnden

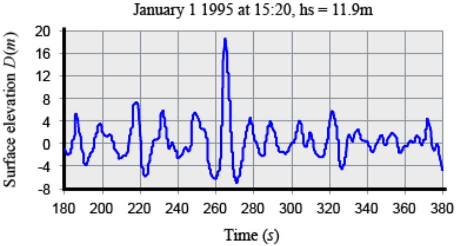
Rogue waves:

Events of extreme ocean surface elevation.

- Mechanism not fully understood
- Probably caused by **nonlinear amplification** (*modulational instability*) out of a background of (smaller) waves
- Probability density function unknown.

Goal: Estimate **tails** of the distribution

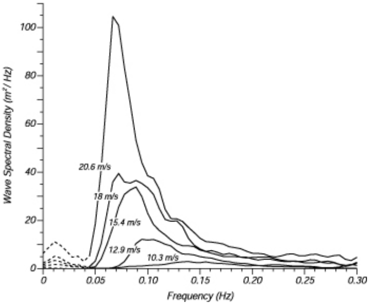
Tanker “Stolt Surf” in 1977, New York Times



“Draupner Wave”, Jan 1, 1995

Two basic ingredients:

- **Random data** from **observations** as input (*prior*)
- Accurate **dynamical system** to extrapolate output (*posterior*)



Wave height observation

Wave channel, TU Hamburg

Estimate **probabilities** of **extreme events** via **large deviation theory**

- The way rare events occur is often predictable — it is dominated by the *least unlikely* scenario — which is the essence of LDT
- Calculation of the least unlikely scenario (maximum likelihood pathway, MLP) reduces to a deterministic optimization problem

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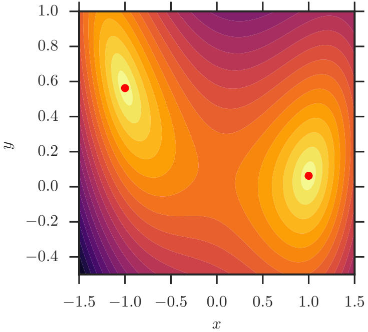

- Simple example: gradient systems (navigating a potential landscape), transitions between local energy minima happen through minimum energy paths (mountain pass transition)

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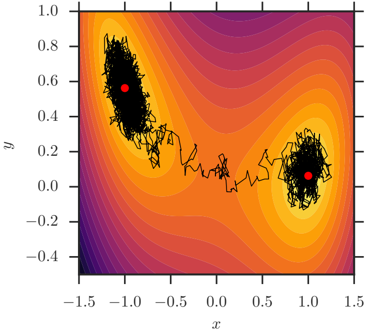

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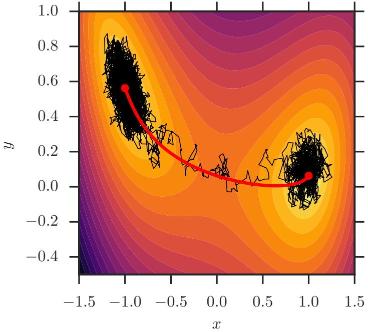

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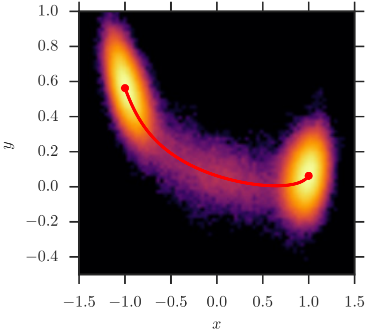

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Consider the S(P)DE

$$dX^\varepsilon(t) = b(X^\varepsilon(t))\,dt + \sqrt{\varepsilon}\sigma(X^\varepsilon(t))\,dW(t)$$

Then, the LDT **rate function** is

$$\mathcal{I}_T(\phi) = \tfrac{1}{2}\int_0^T \left|\sigma(\phi)^{-1}\left(\dot{\phi} - b(\phi)\right)\right|^2 dt = \tfrac{1}{2}\int_0^T \mathcal{L}(\phi,\dot{\phi})\,dt$$

The probability that $\{X^\varepsilon(t)\}_{t\in[0,T]}$ is close to a path $\{\phi(t)\}_{t\in[0,T]}$ is

$$\mathcal{P}\left\{\sup_{0\le t\le T}|X^\varepsilon(t) - \phi(t)| < \delta\right\} \asymp \exp\left(-\varepsilon^{-1}\mathcal{I}_T(\phi)\right) \text{ for } \varepsilon \to 0$$

The problem is reduced to a **minimization** problem

$$\mathcal{P}\left\{F(X^\varepsilon(T)) = z | X^\varepsilon(0) = x\right\} \asymp \exp\left(-\varepsilon^{-1} \inf_{\phi:\phi(0)=x, F(\phi(T))=z} \mathcal{I}_T(\phi)\right)$$

Minimizer ϕ^* is called **instanton**

Example: Ornstein-Uhlenbeck

Ornstein-Uhlenbeck process

$$du = b(u)\,dt + dW\,, \quad b(u) = -\gamma u\,, \quad \gamma > 0\,.$$

Consider extreme events with $u(T) = z$ (so $F(u) = u(T)$).

The **instanton** is

$$u^*(t) = z e^{\gamma(t-T)} \left(\frac{1 - e^{-2\gamma t}}{1 - e^{-2\gamma T}} \right)\,,$$

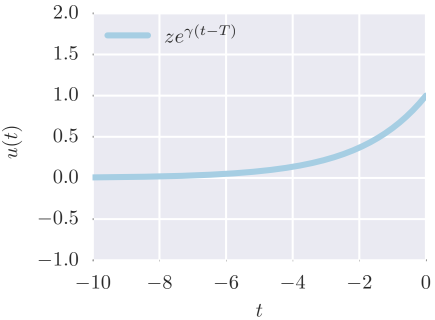
obtained from **constrained optimization**

$$\inf_{\{u_t\} \in \mathcal{U}_z} \mathcal{I}_T(z) = \inf_{\{u_t\} \in \mathcal{U}_z} \tfrac{1}{2} \int_0^T |\dot{u} + \gamma u|^2\,dt$$

over the set

$$\mathcal{U}_z = \Big\{ \{u_t\} \;\Big|\; F(u_T) = z \Big\}$$

Yields **optimal fluctuations** to realize event.



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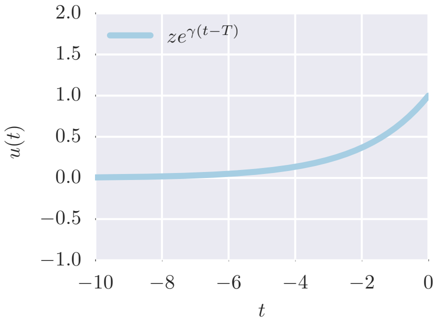

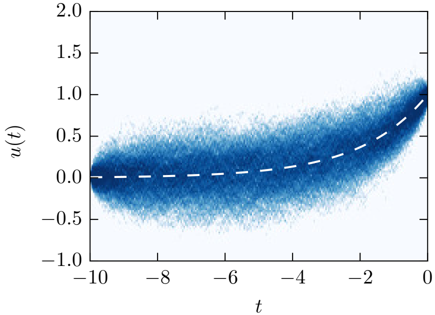

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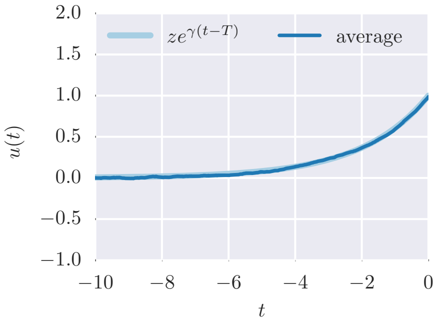

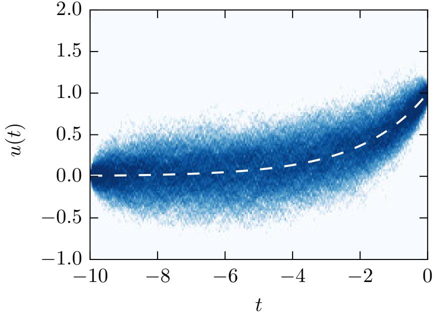

Rogue waves in MNLS

Evolution of a narrow-banded, uni-directional **wave envelope** $u(t,x)$ on the surface of a fluid with **infinite depth**:

modified nonlinear Schrödinger equation (MNLS) or *Dysthe's* equation

$$\partial_t u + \tfrac{1}{2}\partial_x u + \tfrac{i}{8}\partial_x^2 u - \tfrac{1}{16}\partial_x^3 u + \tfrac{i}{2}|u|^2 u + \tfrac{3}{2}|u|^2\partial_x u + \tfrac{1}{4}u^2\partial_x u^* - \tfrac{i}{2}|\partial_x||u|^2 = 0$$

on domain $x \in \Omega$ with **sea surface elevation** $\eta(t,x) = \mathrm{Re}(u(t,x)e^{i(k_0 x - \omega_0 t)})$

- Equation for $u(t,x)$ is deterministic
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Central assumption

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Key difference: LDP on **initial conditions** instead of stochastic dynamics

$$\partial_t u(t) = b(u(t)), \qquad u(0) = u_0 \sim \mu_0 \asymp \exp(-\varepsilon^{-1} G(u_0))$$

Given **observable** $F(u(T))$, we are interested in

$$P_T(z) = \mathcal{P}\{F(u(T)) \geq z\}\,,$$

and extreme events via **LDP** for its density $\rho_T(z) = dP_T(z)/dz$,

$$\rho_T(z) \asymp \exp(\varepsilon^{-1} \mathcal{I}_T(z))\,.$$

Define **cumulant generating function**

$$\epsilon \log \mathbb{E}_{\mu_0} e^{\lambda \epsilon^{-1} F(u(T))} \sim \sup_{u_0 \in \mathcal{U}} \{\lambda F(u(T)) - G(u_0)\} \equiv S_T(\lambda)$$

Then, by the *Gärtner-Ellis* theorem, $\mathcal{I}_T(z)$ is the Fenchel-Legendre transform of $S_T(\lambda)$,

$$S_T(\lambda) = \sup_{z \in \mathbb{R}} (\lambda z - \mathcal{I}_T(z))$$

and therefore

$$\mathcal{I}_T(z) = \inf_{u_0 : F(u(T)) = z} G(u_0)$$

Computational aspects

Define the **Jacobian** $J(t, u_0) = \dfrac{\delta u(t)}{\delta u_0}$ where $u(0) = u_0\,.$

- Solve **constrained optimization** through descent,

$$\frac{\delta S_T}{\delta u_0} = \frac{\delta G(u_0)}{\delta u_0} - \lambda J^T(T, u_0)\frac{\delta F}{\delta u}$$

- The Jacobian evolves according to

$$\dot{J}(t, u_0) = \frac{\delta b(u(t, u_0))}{\delta u} J(t, u_0), \qquad J(0, u_0) = Id$$

- Repeat calculation for various λ to estimate $S_T(\lambda)$, and thus $\mathcal{I}_T(z)$ and **probability** $P_T(z)$.

- Instanton u_0^* gives the **most likely event** leading to **rogue wave** of height z, where

$$u_0^* = \underset{u_0 \in \mathcal{U}}{\operatorname{argmax}} \{G(u_0) - \lambda F(u(T, u_0))\}$$

Large deviations for random initial data: Rogue waves

For **ocean surface waves** described by MNLS:

$$\partial_t u + \frac{1}{2}\partial_x u + \frac{i}{8}\partial_x^2 u - \frac{1}{16}\partial_x^3 u + \frac{i}{2}|u|^2 u + \frac{3}{2}|u|^2\partial_x u + \frac{1}{4}u^2\partial_x u^* - \frac{i}{2}|\partial_x||u|^2 = 0$$

- Initial distribution **Gaussian** and approximates JONSWAP spectrum,

  $$G(u_0) = \frac{1}{2}\langle u_0, C^{-1}u_0\rangle$$

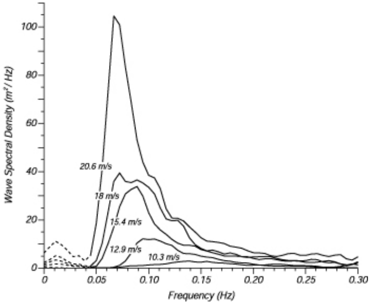
  (but other forms are possible)

- Pick as **observable**

  $$F(u(T)) = \max_{x\in\Omega}|u(T,x)| = \max_{x\in\Omega}\eta(T,x)$$

  the maximum sea surface elevation at final time $t = T$.

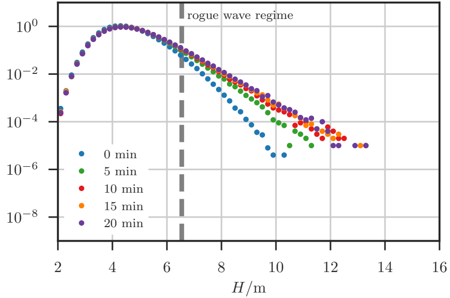
JONSWAP spectrum



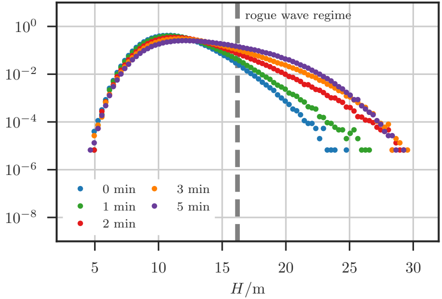
G. Dematteis, T. Grafke, E. Vanden-Eijnden, PNAS 2017 *(to appear), arXiv:1704.01496*

Large deviations for random initial data: Rogue waves

rough sea ($H_s = 3.3$m, BFI $= 0.34$)



high sea ($H_s = 8.2$m, BFI $= 0.85$)



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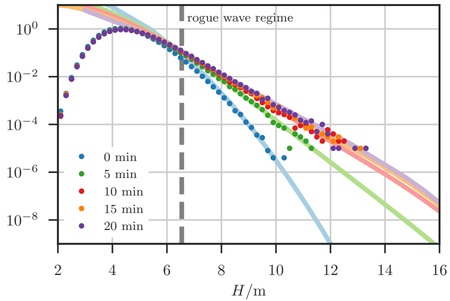

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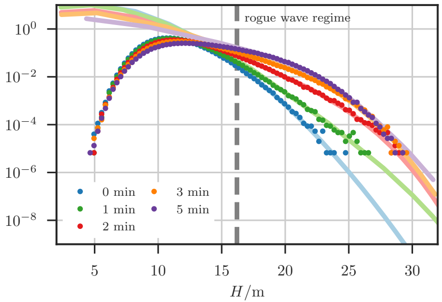

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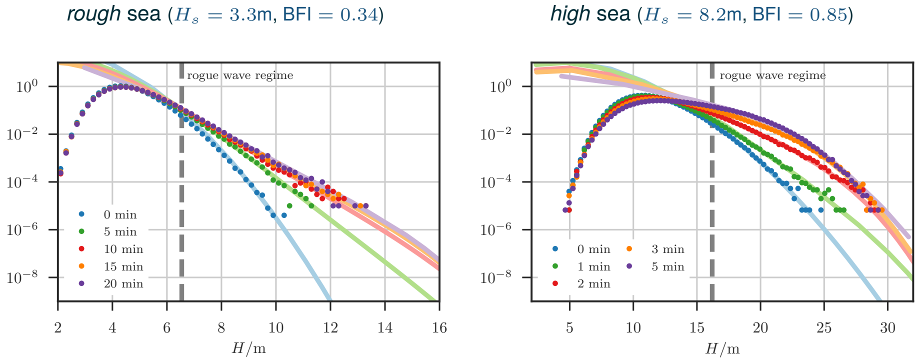

Comparison between **Monte Carlo** (dots) and **LDT** (lines):

- LDT is able to **predict tails** of **rogue wave** distribution.
- **PDF converges** for $T \to 20$min (left) and $T \to 5$min (right).
  (corresponds to timescales associated with **modulational instability**)
- This is the timescale after which the initial distribution is converged to the **invariant measure**, and the **extreme event tails** are correctly predicted.

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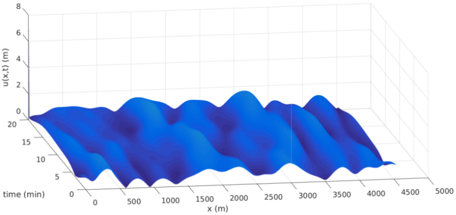
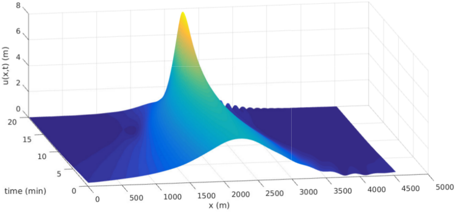
Rogue wave instantons



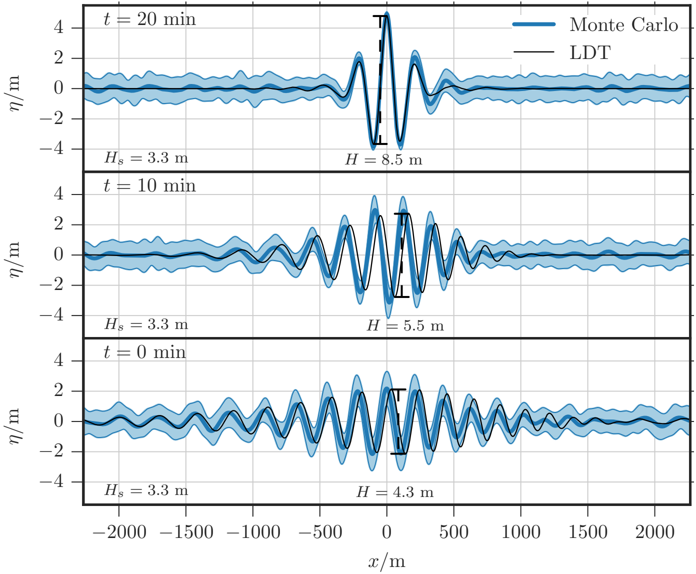
(random initial condition: standard event)



(optimized initial condition: instanton)

- Evolution is purely **deterministic**
- Only operation done here is **pick initial condition**, either at **random** (left) or **optimized** (right)

Large deviations for random initial data: Rogue waves



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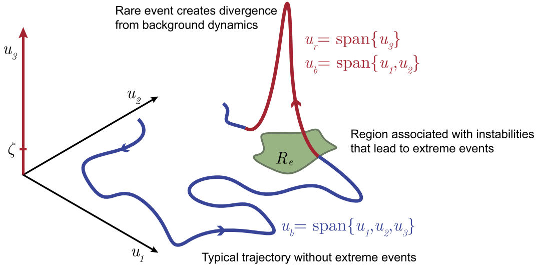
Intermittency and extreme events

- Consistent with picture but forward by Sapsis *et al.* for **intermittency** in dynamical systems
- Assumption: extreme excursions occur when dynamics hit **pockets of instability**
- Estimate **pdf tail** by measuring probability of pockets with respect to the invariant measure of background dynamics



Mohamad, Cousins, & Sapsis, J. Comp. Phys. 322 (2016) 288–308

Our approach formalizes this concept within the realm of **large deviation theory** and gives tools to explicitely compute

- tail probabilities and expectations for observables
- most likely occurrence of extreme events

> Accurate knowledge of the core distribution permits prediction of its tail from determinstic dynamics!

- Initial Gaussian distribution converges to limiting distribution for large times

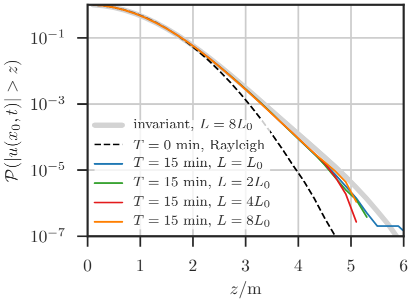
  (prior → posterior)

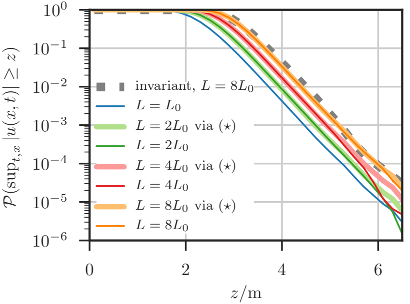
- Can be used to compute probability of extremes in spatio-temporal domain via **boxing argument**

$$\mathcal{P}(\sup_{(t,x)\in\mathcal{D}} |u(t,x)| \geq z) \sim 1-(1-\mathcal{P}(|u \geq z|))^{N_\mathcal{D}} \qquad (\star)$$

  with $N_\mathcal{D} = |\mathcal{D}|/(\lambda_c \tau_c)$

- Extreme events no longer **rare** if domain large enough





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Statistics on large domains

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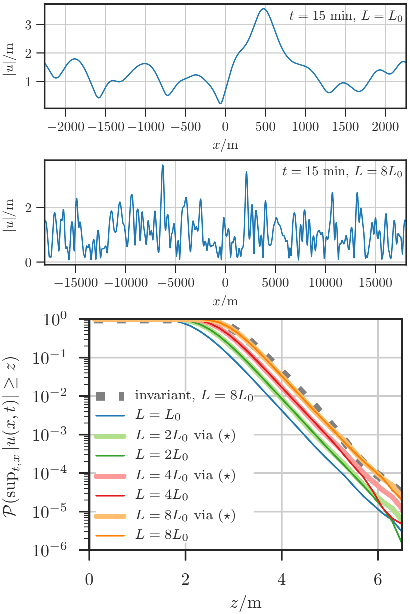

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Comparison to Peregrin soliton

$$u(t,x) = U_i e^{-it/T_{\mathsf{nl}}} \left(\frac{4(1 - 2it/T_{\mathsf{nl}})}{1 + 4\left(t/T_{\mathsf{nl}}\right)^2 + 4\left(x/L_{\mathsf{nl}}\right)^2} - 1 \right), \quad T_{\mathsf{nl}} = \frac{2}{U_i^2}, \quad L_{\mathsf{nl}} = \tfrac{1}{4}\sqrt{T_{\mathsf{nl}}} = \frac{\sqrt{2}}{4U_i},$$

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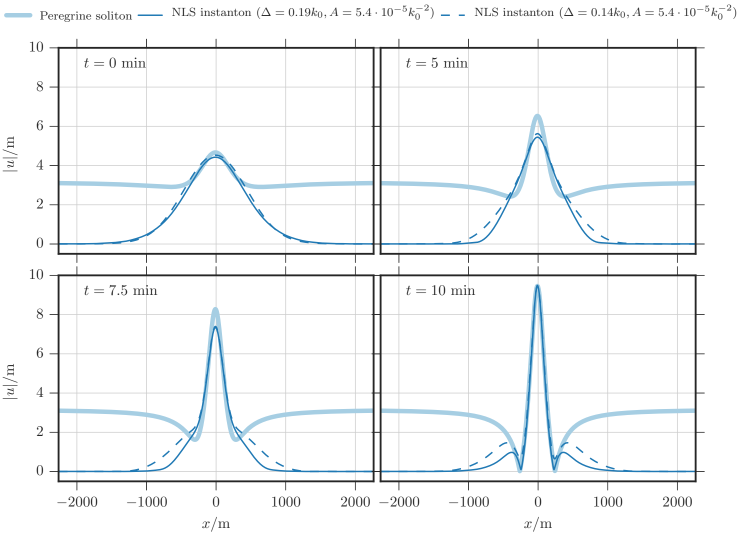

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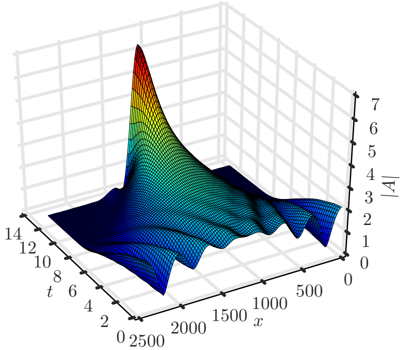
- Twist on LDT: deterministic system (no stochastic forcing) with **random initial data** ($\sim$ JONSWAP)
- Extreme events occurring via **instability** of the determistic dynamics (modulational instability)
- LDT allows to estimate **probability** and **mechanism** of occurrence of rogue waves in MNLS
- Initial distribution plays a role of the **prior distribution** in Bayesian inference. Extreme event information added via short time dynamics to sample the **posterior distribution**.



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